$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho = -\frac{k}{R^2}$$

What does this mean?

If **k=0**, then $\dot{R}$ is always positive, and the expansion continues at an ever slowing pace (since rho is dropping). This is called a **critical or flat universe**.

If **k>0**, $\dot{R}$ is initially positive, but will reach a point where it goes to zero. Expansion stops, gravity wins, and the universe then starts to collapse. This is a **closed universe**.

If **k<0**, $\dot{R}$ is always positive, and never goes to zero -- expansion always continues. This is an **open universe**.

The Friedmann Equation

So that was the equation for Newtonian expansion. It surely can't look anything like the real thing, can it?

Actually, it can. Solving the Einstein field equations for an isotropic, homogeneous Universe yields a similar form for the **dynamics equation**: which is what we had before, but with an additional term (Λ) called the "cosmological constant" expressing the effects of "dark energy" (the vacuum energy density of space?).

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}G\rho + \frac{1}{3}\Lambda c^2$$

Solving this we get the equation governing the expansion of space:

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2 = -\frac{kc^2}{R^2}$$

where k now corresponds to the curvature of space.

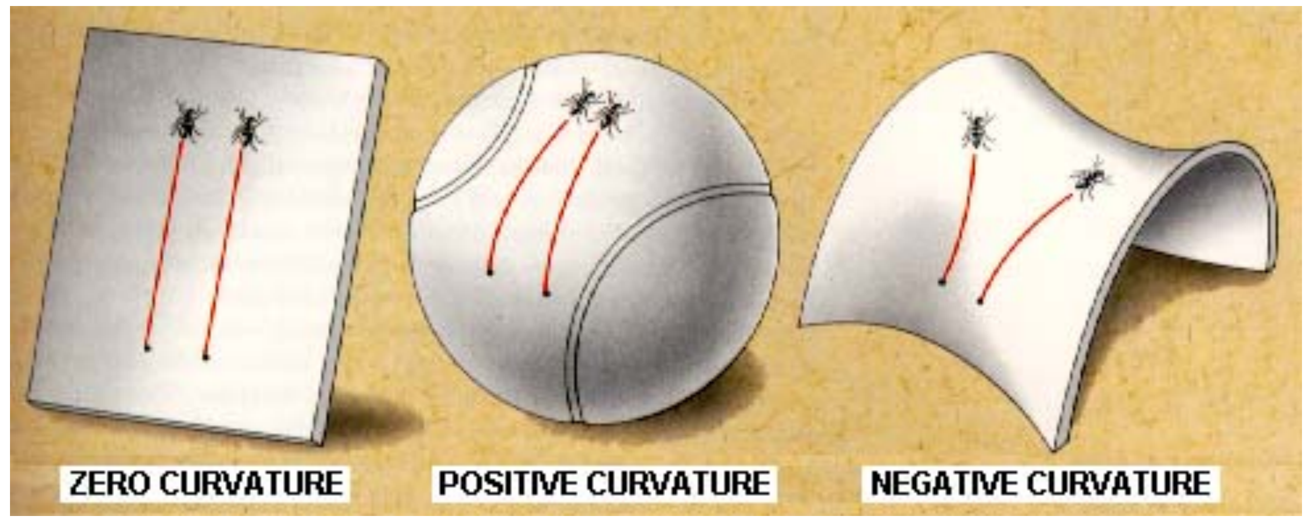
Given different values for ρ, Λ, and k, we can solve this differential equation to get the detailed form of R(t). But first, let's unpack this conceptually.

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2 = -\frac{kc^2}{R^2}$$

1. The Universe can accelerate
Look at the dynamics equation and note that if Λ is positive, it counteracts the deceleration due to gravity and therefore can accelerate the expansion of the universe.

2. Space is curved
Think of ants crawling around on an expanding balloon.
We have three options for curvature:

- k=1: positive curvature (like a ball)
- k=0: zero curvature (flat like paper)
- k=-1: negative curvature (saddle-like)



But remember, this is only an analogy to what is really going on with curved spacetime. Like all analogies, it can only be pushed so far. In non-Euclidian space, distances and sizes do non-intuitive things. *This allows us to make observational tests of cosmology.*

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2 = -\frac{kc^2}{R^2}$$

3. Density (rho) can take on different forms, and the universe behaves differently under each one.
We made the assumption of a matter dominated Universe (so that $\rho \sim R^{-3}$). This was not always true in the Universe. ρ is actually the inertial mass density of the matter and radiation in the Universe. If the energy density in the universe is given by U, we can convert that into an equivalent matter density

$$\rho_{\rm rad} = U/c^2$$

and relate it to the photon density like this

$$U = \rho_{\rm phot} E_{\rm phot} \sim R^{-4}$$

So if we trace the history of the Universe back far enough, at some point the energy density of radiation exceeds that of matter, and we get to the **radiation dominated era**. At this point, the expansion dynamics are different, driven by radiation rather than gravity. This happens at a scale factor of $R=10^{-3}$ - 10^{-4}, so we can ignore it for the purposes of observational cosmology. But it does change the expansion rate in the early universe...

4. The future of expansion if Λ is really a constant...
The energy density of matter and radiation drops as the universe expands. The density of dark energy does not. Eventually, Lambda wins. In this case, R ~ exp(t) -- exponential expansion.

5. The interplay between expansion history and curvature becomes very complex
Without Λ, spatial curvature and expansion history were locked together. With Λ, that connection is broken. Closed universes can expand forever, open universes can collapse, etc. Life becomes messy!

1. The Hubble Parameter

The Hubble parameter is the normalized rate of expansion:

$$H \equiv \frac{\dot{R}}{R}$$

Note that the Hubble parameter is not a constant! It changes with time. The Hubble constant is the Hubble parameter measured today -- we denote its value by H_0. Best estimates are in the range of H_0 = 65-75 km/s/Mpc. Also note you will often see the parameter h, particularly in distance-dependant quantities (for example, $30h^{-1}$ Mpc). This is usually defined by $h=H_0/100$.

2. The Matter Density Parameter.

Look at the Friedmann equation:

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2 = -\frac{kc^2}{R^2}$$

Rewriting this using the Hubble parameter, and for now set Λ=0:

$$H^2 - \frac{8}{3}\pi G\rho = -\frac{kc^2}{R^2}$$

The Universe is flat if k=0, or if it has a critical mass density (in the absence of Λ) of

$$\rho_{\rm crit} = \frac{3H^2}{8\pi G}$$

We define the **matter density parameter** as $\Omega_M = \frac{\rho}{\rho_{\rm crit}}$

Best measurements for Ω_M are about 0.25 - 0.3, meaning the universe cannot have spatial flatness based on mass alone.

3. The "dark energy" density parameter

We can express a similar density parameter for lambda again by using the Friedmann equation and setting ρ_m=0. We then get

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H^2}$$

Best measurements for Ω_Λ are about 0.7-0.75

4. "Total Omega"

$$\Omega = \Omega_M + \Omega_\Lambda$$

Questions: What is Ω if the Universe is flat? What is Ω if the Universe is accelerating?

Best estimates for total Ω are about 1.0

5. The deceleration parameter

We can make a dimensionless parameter that describes the de/acceleration rate of the Universe's expansion:

$$q = -\frac{R\ddot{R}}{\dot{R}} = \frac{\Omega_M}{2} - \Omega_\Lambda$$

Given values above, q ~ -0.55, in other words the universe's expansion is *accelerating*.

If the universe is expanding, and has a finite age, if we run it backwards we must come to a point in the Universe's history when it was very dense and very hot (this was referred to, deridingly, by Fred Hoyle as the "Big Bang").

In such a hot, dense universe, the mean free path of photons is small, so energy cannot be radiated away. **The particles and photons are thermalized, and produce a black body spectrum.** The early universe was a hot black body characterized by a temperature T. **As the Universe expands and cools, this blackbody radiation maintains it spectral shape, but has a characteristic temperature $T \sim 1/R$**

In the late 1940s and early 1950s, George Gamov suggested that the elements in the Universe were "cooked" via nuclear reactions in the hot Big Bang. Later, Gamov and Alpher showed that actually the Big Bang could not create all the elements, but that the lighter elements, helium in particular, could have formed in this way.
Important constraint: **To make Helium this way, the temperature must have been 10^9 K and the densities ~ 10^{-5} g/cm^3. (*note: higher temps and lower densities than in the Sun*).**

Since density scales as R^{-3}, we can estimate the scale factor of the Universe when this happened (if we have an estimate of the current baryonic density). Since

$$\rho_0 = \rho R^3$$

We have $R \sim (\rho_0/\rho)^{1/3} \sim 3 \times 10^{-9}$

Since this happened at $T=10^9$, and the radiation cools like 1/R, we should have radiation running around the universe with a characteristic temperature of

$$T_0 = RT = (3 \times 10^{-9})(10^9) = 3 \text{ Kelvin}$$

This peaks in the microwave portion of the spectrum.

Gamov and Herman predicted the Universe should be glowing in microwaves