Cosmological Framework

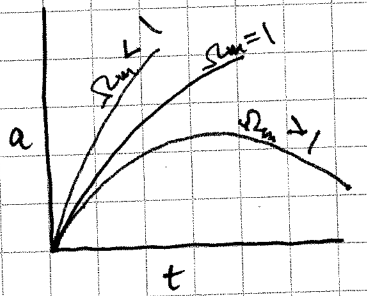
Dark matter halos are thought to form by gravitational collapse of over-dense regions in an otherwise expanding universe.

$a = \frac{1}{1+z}$

A little necessary context:

Friedmann eqn: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$

scale size | mass density | curvature | cosmological constant



a is the scale size of the universe — a dimensionless quantity that encodes the physical separation between comoving coordinate tracers (e.g., galaxies w/ zero peculiar motion)

a(t) is the expansion history of the universe following from the solution of Friedmann's eqn.

Note that the Hubble parameter $H = \frac{\dot{a}}{a}$ is the expansion rate.

H must vary with time; it's current measured value is the misnamed Hubble "constant" $H_0 = \left(\frac{\dot{a}}{a}\right)_0$ measured now at $t = t_0$, the age of the U.

It is also convenient to define the density parameter $\Omega_m = \frac{\rho}{\rho_{crit}}$

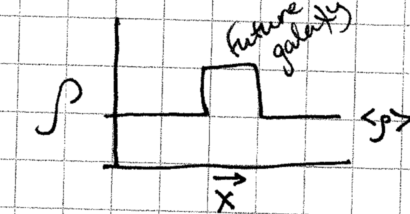
which is the ratio of the actual mass density to the critical density

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$ that defines the over/under between eternal expansion and eventual recollapse.

Note that ρ_{crit} evolves with H, as do ρ & Ω_m.
Only if $\Omega_m = 1$ exactly does it remain 1 for eternity

The Friedmann eqn can be derived from the eqn. of motion of a point on the surface of a uniform expanding sphere, at least in the absence of Λ. It does not depend on scale. So a good first approximation to galaxy formation is to treat the volume that will collapse to form a galaxy as a locally overdense universe with $\Omega_m > 1$.

This is often called a top-hat overdensity



Note that $\vec{x}$ represents all 3 spatial dimensions. The over-dense region is spherical, so Friedmann's eqn applies.

Indeed, as $t \to 0$, $\Omega_m \to 1$, so the curvature and Λ terms may be ignored for the mean $\langle\rho\rangle$.

The solution for the uniform background universe (in which the top-hat is embedded) is

$$\frac{a}{a_0} = \left(\frac{3}{2} H_0 t\right)^{2/3}.$$

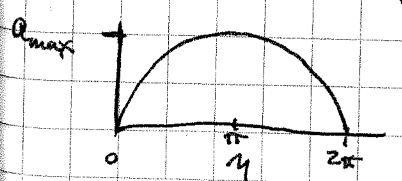
For the top hat, $\Omega_m > 1$, the condition necessary for it to collapse.

But only a little > 1: the initial condition is set by the fluctuations in the CMB when $t \approx 10^5$ yr and $\frac{\delta\rho}{\langle\rho\rangle} \approx 10^{-5}$ — galaxies are thought to arise from the gravitational growth and collapse of these initially tiny over-densities. (Under-densities become voids.)

For $\Omega_m > 1$ the solution is

$$\frac{a}{a_0} = \frac{1}{2}\,\frac{\Omega_{m,0}}{\Omega_{m,0} - 1}\left(1 - \cos\eta\right); \quad H_0 t = \frac{1}{2}\,\frac{\Omega_{m,0}}{(\Omega_{m,0} - 1)^{3/2}}\left(\eta - \sin\eta\right)$$

η is the "development parameter" representing time to recollapse



η runs from 0 to 2π with maximum expansion at $\eta = \pi$

also used in timing argument