The Oort Discrepancy

The Milky Way, like most spiral galaxies,
is a thin, dynamically cold, rotating disk of stars & gas.

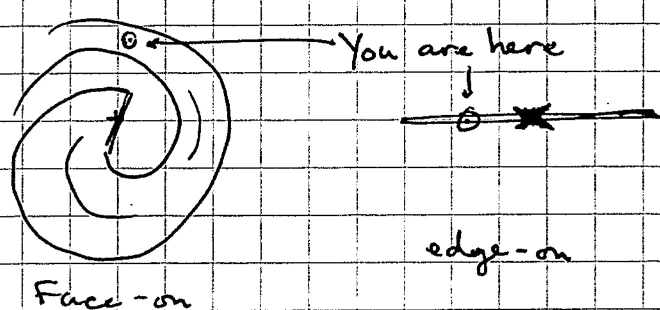
Thin: $R_d : z_0 = 8:1$ is typical, with considerable scatter

Dynamically Cold: $V/\sigma \gg 1$

for the Milky Way, $V \approx 200$ km s^{-1}

$\sigma \approx 20$ km s^{-1}

This is typical. Understanding the stability & persistance of thin spiral disks is a challenge, but the universe is littered with them.



To a decent first approximation, the orbit of a star in the Milky Way is approximately circular, ~~with~~ ~~bar~~ in the plane of the disk, while executing harmonic oscillation in the z-direction.

Oort (1932) considered the z-motions of stars in the solar neighborhood, extending and applying earlier work by Kapteyn and Jeans.

Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho$$

Φ = gravitational potential
ρ = 3D density ($M_\odot\, pc^{-3}$)

In cylindrical coordinates R, θ, z

$$\nabla^2 \Phi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) + \underbrace{\frac{1}{R^2}\frac{\partial^2 \Phi}{\partial \theta^2}}_{\to 0} + \frac{\partial^2 \Phi}{\partial z^2}$$

Assume axis-symmetry, so $\frac{\partial \Phi}{\partial \theta} = 0$. Not really valid if spiral arms are strong (or bars)

integrate once over z

$$\Sigma = \int_{-|z|}^{|z|} \rho\, dz$$

for distributions symmetric in z $\left[\rho(z) = \rho(-z)\right]$

this is equivalent to $\int_0^{|z|}$ times 2,

so $4\pi G\rho \to 2\pi G\Sigma |z|$

assume separable:

radial force $\quad K_R = -\frac{\partial \Phi}{\partial R} = -\frac{V^2}{R}$

vertical force $\quad K_z = \frac{\partial \Phi}{\partial z} = \int \frac{\partial^2 \Phi}{\partial z^2}\, dz$

so now we've eliminated one derivative to have

$$2\pi G\Sigma = |K_z| - |z| \frac{1}{R}\frac{\partial}{\partial R}(R K_R) \qquad \left(R K_R = R \cdot \frac{V^2}{R} = V^2\right)$$

so

$$K_z = 2\pi G\Sigma + \frac{z}{R}\frac{\partial V^2}{\partial R}$$

restoring force to plane

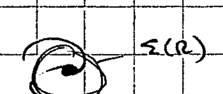
surface density of plane w/ $z < |z|$
depends on disk surface density

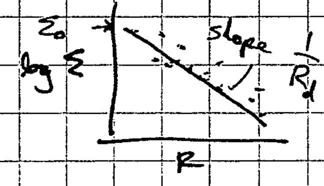
radial force term amounts to square of rotation velocity
also depends on DM halo

By convention, vertical force has been measured at $|z| = 1.1$ kpc, which contains most of the disk mass, but still has stars left to measure

Double exponential model

The azimuthally averaged light profile of spiral galaxies can tolerably be approximated by an "exponential disk"





$$\Sigma(R) = \Sigma_0 e^{-R/R_d}$$

Σ_0 central surface brightness
R_d disk scale length

This describes the 2D face-on distribution of star light.
The z-distribution can also be approximated as exponential, so
In 3D

$$\rho(R, z) = \rho_0 e^{-R/R_d} e^{-z/z_0}$$

We can use this to integrate the Poisson equation and replace K_z with the vertical velocity dispersion σ_z

The result is $\qquad \sigma_z^2 = K \pi G z_0 \Sigma_0$ $\qquad$ Very local approximation
Get cross terms if δR large

where the constant K depends on the mass distribution.
Usually it is taken to be $K = 2$ (~~uniform slab~~) isothermal
or $K = 1.5$ (pure exponential vertical dist)

In general, this applies to all components —
every population of stars (AV, KIII, etc)
as well as gas, and dark matter.
The constant K can be determined numerically for arbitrary $\Sigma(z)$, but for realistic cases $1.5 \leq K \leq 2$

Will return to this with the Jeans equations and the concept of the phase space. These give a different approach to individual tracer populations, as young stars are thinner & dynamically colder than older stellar populations

MEASUREMENTS

There have been many attempts to measure the local surface density of the disk since Oort:

e.g.,

Kuijken & Gilmore 1989
Holmberg & Flynn 2004
Bienaymé et al 2014 — to $z > 2$ kpc!
Bovy & Rix 2013 — $K_{||}$ over large range in radii, not just locally!

numerically, the accounting goes (at the solar circle, $R = R_0$)

$$\text{at } R_0 \quad \left.\begin{array}{l} \Sigma_* = 38\ M_\odot\,pc^{-2} \\ \Sigma_{gas} \doteq 14\ M_\odot\,pc^{-2} \end{array}\right\} \Sigma_b = 52\ M_\odot\,pc^{-2}$$

$$\Sigma_{dyn} \doteq 74\ M_\odot\,pc^{-2} \quad > \Sigma_b$$

That $\Sigma_{dyn} > \Sigma_b = \Sigma_* + \Sigma_g$ is a mass discrepancy.
mostly (entirely?) due to $\frac{\partial V^2}{\partial R}$ term

The dynamics wants more mass than is directly seen.

The difference — $22\ M_\odot\,pc^{-2}$ — is presumably the integrated portion of a quasi-spherical dark matter halo between $-1.1 \le z \le 1.1$ kpc.

For comparison, the local V-band surface brightness is $\sim 27\ L_\odot\,pc^{-2}$. So the mass-to-light ratio of the stars of the Milky Way is a reasonable

$$\Upsilon_*^V = 1.4\ M_\odot/L_\odot. \qquad = \Sigma_*/\Sigma_V = 38\ M_\odot\,pc^{-2} / 27\ L_\odot\,pc^{-2}$$

The discrepancy is due to the additional force provided by a quasi-spherical dark matter halo, ~~the~~ which contributes through the radial force term in the Poisson equation. There is no extra dark matter restricted to the disk (3D Oort limit).

Disk Stability

Ostriker & Peebles (1973)

- very early, rather primitive numerical simulations
- showed that thin, cold, rotating disks are unstable to the growth of $m=2$ modes (the bar instability).
- this problem remains unresolved (arXiv:1601.03406).

O & P solution was to embed spiral disks in a deep potential well – a dark matter halo.

Split kinetic energy into rotational T and pressure Π components

so

$$K = T + \tfrac{1}{2}\Pi \qquad (T \text{ all in plane of disk. } \Pi \text{ not})$$

$$K = \tfrac{1}{2}|W|$$ Virial equilibrium condition
kinetic energy = half the potential energy

O & P found stability was achieved when

$$t \lesssim 0.14 \qquad t \equiv \frac{T}{|W|}$$ the rotational component of the KE normalized to the P.E.

This is crude & almost certainly wrong in detail, but was important early evidence for Dark Matter

e.g.: the Milky Way has $V_c \approx 220 \text{ km s}^{-1}$ $\quad T \sim V_c^2$
$\sigma \approx 20 \text{ km s}^{-1}$ $\quad \Pi \sim \sigma^2$

working those numbers, $t \approx 0.45$ [0.5 is the maximum]

Such a cold disk should be wildly unstable, yet spiral galaxies persist over a Hubble time

Dark Matter halos provide stability (maybe)
provided $\Pi \gg T$: they must be dynamically hot, quasi-spherical, and dominate the mass

Other stability criteria

Global stability against non-axis-symmetric perturbations

$$X_m = \frac{\kappa^2 R}{2\pi m G \Sigma}$$

Goldreich & Tremaine '78, '79
Toomre 1981

m is mode number of perturbation – $m = 2$ for a bar or grand design spiral

κ is the epicyclic frequency

Σ the surface density

The higher the surface density Σ, the less stable the disk

The precise stability criterion is determined numerically and depends on the potential

$X > 3$ required for stability for a flat rotation curve

$X > 1$ suffices for a rising rotation curve

Lowest modes m suppressed first (Athanassoula et al 1987)

So a very high surface density disk is unstable
a merely high " " " can have $m = 2$ modes (bars, grand design spirals)
a low " " " will have $m = 2$ suppressed.

It is tempting to conclude the grand design spirals are nearly maximal – high enough surface density to drive $m = 2$ modes
Perhaps flocculent spirals are lower Σ so $m = 2$ suppressed

Bars & spiral arms are the natural consequence of disk self-gravity. If the halo becomes too dominant, all modes get suppressed.

The amplitudes of modes are usually expressed as Fourier components

$$A_m = \frac{1}{N}\sum_{j=1}^{N} e^{i[m\theta_j + p\ln(r_j)]}$$

A is the amplitude of mode m for N stars

m = 2 gives a bar for p = 0
" " grand design spiral for p > 0

p is the "pitch angle" of a "logarithmic spiral"

LOCAL disk stability criterion "Toomre Q"
(Toomre 1964)

$$Q = \frac{\sigma_r K}{3.36\, G \Sigma}$$

σ_r = radial velocity dispersion
Σ = local disk surface density
K = epicyclic frequency

3.36 Constant chosen so that stability occurs for $Q \gtrsim 1$

Numerically, $Q \approx 1.4$ in solar neighborhood
Instability $Q \lesssim 1$ sometimes thought to be a criterion for star formation.

EPICYCLIC FREQUENCY

$$K^2 = \frac{\partial^2 \Phi}{\partial R^2} = R\frac{\partial \Omega^2}{\partial R} + 4\Omega^2$$

where $\Omega = \frac{V}{R}$ is the orbital frequency.

K is the frequency with which a star oscillates (in the plane) about the "guiding center" of a closed (circular) orbit.