

Potential - Density Pairs

BT chapter 2

$\Phi - \rho$ combinations that satisfy the Poisson equation $\nabla^2 \Phi = 4\pi G \rho$

Precious few that are analytical.

In general, one must solve Poisson numerically for arbitrary ρ .

Bear in mind: the gravitational potential $\Phi \sim v^2$
 $a \sim \frac{v^2}{R} \approx -\frac{\partial \Phi}{\partial R}$

Plummer model:

first used for globular clusters, has also been used for fixed halo in merger simulation
E-galaxies, clusters of galaxies.
A good "general purpose" analytic spherical potential

Potential:

$$\Phi_P = -\frac{GM}{\sqrt{r^2 + b^2}}$$

$$\nabla^2 \Phi_P = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi_P}{dr} \right) = \frac{3GM b^2}{(r^2 + b^2)^{5/2}} = 4\pi G \rho_P$$

so $\rho_P(r) = \left(\frac{3M}{4\pi b^3} \right) \left(1 + \frac{r^2}{b^2} \right)^{-5/2}$

In limit

$$\begin{array}{l} r \ll b \\ (r \rightarrow 0) \\ r \ll b \\ (r \rightarrow \infty) \end{array}$$

$$\rho_P \rightarrow \frac{3M}{4\pi b^3} \sim \text{constant}$$

$$\rho_P \rightarrow \frac{3M b^2}{4\pi r^5} \sim r^{-5}$$

so total mass finite

Plummer a common "first-guess" in modelling spherical systems.

Is one of the polytrope family

Modified inertia

Potential - Density Pairs

Kuzmin disk

special case of Toomre Model (#1)

now consider

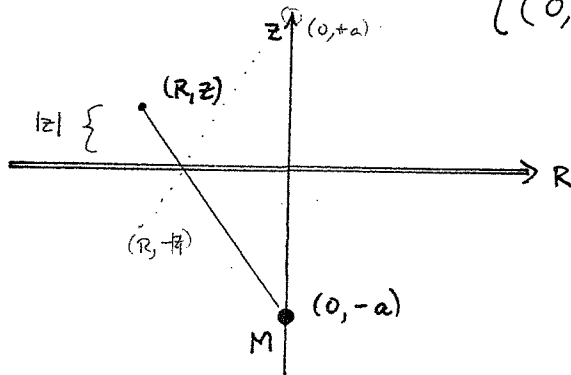
axis-symmetric (cylindrical, not spherical) potential

$$\Phi_K(R, z) = - \frac{GM}{\sqrt{R^2 + (a + |z|)^2}}$$

R now in x-y plane

Note that outside the plane, Φ_K is identical to that of a point mass located at

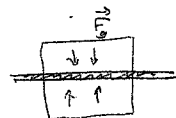
$$(R, z) = \begin{cases} (0, a) & \text{for } z < 0 \\ (0, -a) & \text{for } z > 0 \end{cases}$$



Hence, $\nabla^2 \Phi_K = 0$ for $z \neq 0$

Apply Gauss's theorem to plane to get

$$\Sigma_K(R) = \frac{aM}{2\pi(R^2 + a^2)^{3/2}}$$



$$\int \nabla \phi \cdot d\mathbf{l} = 4\pi GM$$

$$\nabla \phi = 2\pi G \Sigma$$

for $z \rightarrow 0$

$$\Sigma = \frac{1}{2\pi G} \left. \frac{\partial \phi}{\partial z} \right|_{z=0}$$

This is an infinitesimally thin disk

$$\Sigma_K(z) = \delta(z)$$

not physical, but often a good approximation
(disk systems are, by their nature, thin)

Toomre's models n

Generalize Kuzmin disk by constructing series of linear combinations of Φ - p pairs;

e.g., by $\frac{\partial^{n-1} \Phi_K}{\partial a^2}$ with $a^2 = 2nR_T^2$

Doing this gives

$$\sum_K^n(R) = \sum_0 \left(1 + \frac{R^2}{2nR_T^2}\right)^{-(n+\frac{1}{2})}$$

in the limit $n \rightarrow \infty$, becomes Gaussian

$$\sum_K^\infty(R) = \sum_0 e^{-\frac{R^2}{2R_T^2}}$$

...

Can play similar generalization game with

Miyamoto-Nagai potential (eqs. 2-53 B&T)

Potential of Exponential Disk

The disk components of spiral galaxies
(i.e., THE STARS) are generally exponential in nature:

$$\Sigma(R) = \Sigma_0 e^{-R/R_d}$$

(integrated mass distribution)

$$M(R) = 2\pi \int R \Sigma(R) dR$$

$$= 2\pi \Sigma_0 R_d^2 \left[1 - e^{-R/R_d} \left(1 + \frac{R}{R_d} \right) \right]$$

Hankel transform (like Fourier transform, but for cylindrical things)

$$S(k) = -2\pi G \int_0^{\infty} \underbrace{J_0(kR)}_{\text{cylindrical Bessel fun of order zero}} \Sigma(R) R dR$$

For exponential disk

$$S(k) = - \frac{2\pi G \Sigma_0 R_d^2}{[1 + (kR_d)^2]^{3/2}}$$

then

$$\Phi(R, z) = -2\pi G \Sigma_0 R_d^2 \int_0^{\infty} \frac{J_0(kR) e^{-k|z|}}{[1 + (kR_d)^2]^{3/2}} dk$$

disk mass $M_d = 2\pi \Sigma_0 R_d^2$

actually doing this (Freeman 1970)

gives

$$\Phi(R, 0) = -\pi G \Sigma_0 R \left[I_0(y) K_1(y) - I_1(y) K_0(y) \right]$$

which gives
the rotation curve

$$y \equiv \frac{R}{2R_d}$$

$$V_c^2(R) = R \frac{\partial \Phi}{\partial R} = 4\pi G \Sigma_0 R_d y^2 \left[I_0(y) K_0(y) - I_1(y) K_1(y) \right]$$

peaks @ $2.2 R_d$

$I_n(y)$ = modified Bessel fun of first kind

$K_n(y)$ = " " " " " second "

<http://www.astro.umd.edu/~nssm/expdisk.bess>

NOTE: $V(R)$ from $\Sigma(R)$ "straight forward", in practice perform numerical Hankel transform of real observed $\Sigma(R)$
 $\Sigma(R)$ from $V(R)$ → errors blow up in your face

BT 2.6

Logarithmic Potentials

Potentials we've discussed so far arise from finite mass distributions, and so have Keplerian rotation behavior $v_c(r) \sim r^{-1/2}$ at large r .

But real spiral galaxies have flat rotation curves! $v_c \rightarrow \text{const}$ for large r

$$\text{acceleration} = \frac{v_c^2}{R} = -\frac{\partial \Phi}{\partial R} \propto \frac{1}{R}$$

$$\rightarrow \Phi \propto \ln R + \text{const}$$

motivates form

$$\Phi_L = \frac{1}{2} v_0^2 \ln \left(R_c^2 + R^2 + \frac{z^2}{q_\Phi^2} \right) + \text{const}$$

R_c and v_0 are constants (size scale & "depth" of potential)

$$q_\Phi \leq 1$$

Equipotential surfaces are ellipses with axis ratio q_Φ

Density distribution complex but analytic (BT eqn 2-54b)

show transparency (BT Fig 2-8) showing equidensity contours

Note that formally $\rho < 0$ when $q_\Phi = 0.7$ and $|z| \geq 7R_c$

Weird density distribution due to "disk + halo" mass distributions

new BT
eq. 2.71c

BT 2.3.2

Logarithmic potentials are often a good approximation for dark matter halos in an empirical sense, but they are NOT what is predicted by structure formation simulations.