Cosmological Overdensities & Enclosed Mass: Relating $\mathbf{R}_\Delta$, $\mathbf{M}_\Delta$, and $\mathbf{V}_\Delta$

It is conventional in cosmology to refer to structures by the density contrast they represent with respect to the critical density of the universe. The mass enclosed within a radius encompassing the over-density Δ is

$$M_\Delta = \frac{4\pi}{3}\Delta\rho_{crit}R_\Delta^3.$$

With the definition of critical density

$$\rho_{crit} = \frac{3H_0^2}{8\pi G},$$

this becomes

$$M_\Delta = \frac{\Delta}{2G}H_0^2 R_\Delta^3.$$

By the same token, the circular velocity of a tracer particle at R_Δ is

$$V_\Delta^2 = \frac{GM_\Delta}{R_\Delta}.$$

Consequently,

$$M_\Delta = (\Delta/2)^{-1/2}(GH_0)^{-1}\, V_\Delta^3.$$

The choice of reference Δ is somewhat arbitrary. For the current best-fit ΛCDM cosmology, the "virial" radius within which the mass has had time to settle down occurs around $\Delta \approx 100$. (The exact virial value of Δ depends weakly on the the cosmological parameters. The definition of virial radius, while formally meaningful, is nevertheless somewhat arbitrary as the mass profiles of dark matter halos merge smoothly into their surroundings.) For a Hubble constant $H_0 = 72\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$,

$$M_{100} = (4.6 \times 10^5\ \mathrm{km^{-3}\,s^3\,M_\odot})V_{100}^3.$$

for $\Delta = 200$, as conventionally (if not quite correctly) used,

$$M_{200} = \left(3.3 \times 10^5 \left(\frac{\mathrm{km}}{\mathrm{s}}\right)^3 M_\odot\right) V_{200}^3$$

NFW Density Profile

Halo only

Navarro, Frenk & White (1996, ApJ, **462**, 563) define the spherical density profile

$$\rho(r) = \frac{\rho_s r_s^3}{r(r+r_s)^2}.$$

This integrates to a mass profile

$$M(r) = 4\pi\rho_s r_s^3 \left[\ln(1+x) - \frac{x}{1+x}\right] = M_* \left[\ln(1+x) - \frac{x}{1+x}\right],$$

where $x = r/r_s$ and $M_* = 4\pi\rho_s r_s^3$. For small x,

$$M(x) \simeq M_* \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{3x^4}{4}\right] \simeq 2\pi\rho_s r_s r^2.$$

The gravitational potential is the work done

$$\Phi(r) = -\int_r^\infty \frac{GM(r')}{r'^2} dr'.$$

Inserting the above expression for $M(r)$, we obtain

$$\Phi(r) = -\frac{GM_*}{r_s} \frac{\ln(1+x)}{x}.$$

Notice that this does tend to zero as $r \to \infty$ even though the total mass is logarithmically divergent. At small radii

$$\Phi(r) \simeq -\frac{GM_*}{r_s} \left[1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4}\right]$$

The radial acceleration is

$$-\frac{d\Phi}{dr} = \frac{GM_*}{r_s^2} \left[\frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2}\right],$$

and the circular speed is

$$V^2(r) = \frac{GM_*}{r} \left[\ln(1+x) - \frac{x}{1+x}\right].$$

Since

$$\kappa^2 = 2\Omega\left(\Omega + \frac{dV}{dr}\right) \equiv \frac{2V^2}{r^2} + \frac{1}{r}\frac{dV^2}{dr},$$

we find

$$\kappa^2(r) = \frac{GM_*}{r^3} \left[\ln(1+x) - \frac{x}{1+x} + \frac{x^2}{(1+x)^2}\right].$$

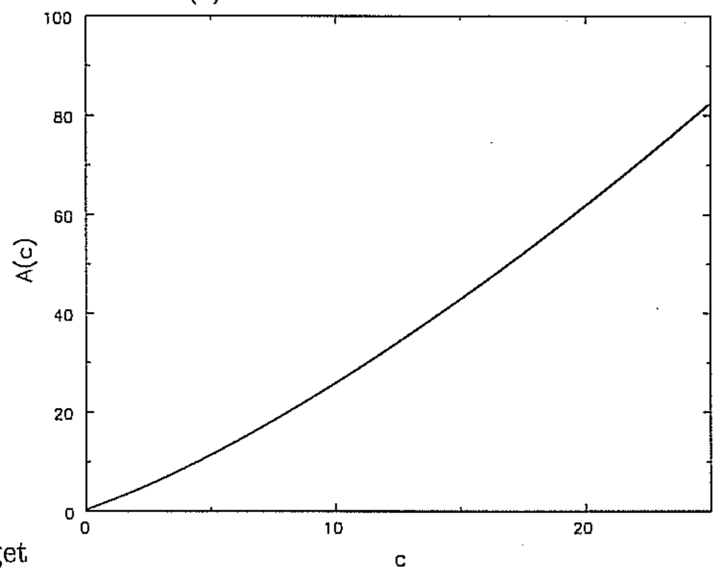
NFW introduce the "virial radius" r_{200}, inside of which the *average* density is $200\rho_{\rm crit}$, with $\rho_{\rm crit} = 3H_0^2/(8\pi G) = 1.88 \times 10^{-29}h^2$ g cm^{-3} $= 2.76 \times 10^{-7}h^2$ M$_\odot$ pc^{-3}. They further define $c = r_{200}/r_s$. Now

$$V_{200}^2 = \frac{GM_{200}}{r_{200}} = \frac{G}{cr_s}200\frac{4\pi}{3}c^3r_s^3\frac{3H_0^2}{8\pi G}, \qquad \text{or} \qquad V_{200} = 10cr_sH_0,$$

and

$$\frac{\rho_s}{\rho_{\rm crit}} = \frac{200c^3}{3[\ln(1+c) - c/(1+c)]} \equiv \frac{200}{3}\left[A(c)\right]^2.$$

For small values of c we have $[A(c)]^2 \simeq 6c/(3-2c)$. (For those interested in trivia, $\rho_s = \rho_{\rm crit}$ when $c \approx 3/402$.) The function $A(c)$ looks like:



From this, we also get

$$V(x) = V_{200}\left[\left(\frac{c}{x}\right)\frac{\ln(1+x) - x/(1+x)}{\ln(1+c) - c/(1+c)}\right]^{1/2},$$

which has a maximum at $x \simeq 2.162582$ with the value

$$V_{\rm max} \simeq 0.465V_{200}\left[\frac{c}{\ln(1+c) - c/(1+c)}\right]^{1/2} = \frac{0.465V_{200}}{c}A(c) = 4.65r_sH_0A(c).$$

The average density inside r_s is

$$\bar{\rho}_1 = \frac{3V^2(1)}{4\pi Gr_s^2} = \frac{3V_{200}^2}{4\pi Gr_s^2}\frac{A^2(c)}{c^2}[\ln 2 - 1/2] = \frac{3 \times 0.193A^2(c)}{4\pi G}100H_0^2.$$

In terms of the critical density, this is

$$\frac{\bar{\rho}_1}{\rho_{\rm crit}} = \frac{3 \times 19.3A^2(c)}{4\pi G}H_0^2 \times \frac{8\pi G}{3H_0^2} = 38.6A^2(c),$$

which is less than ρ_s because $\rho(r_s) = \rho_s/4$ – in fact, $0.58\rho_s \approx \bar{\rho}_1 \approx 2.32\rho(r_s)$.

Einasto halo

A better fitting fcn for simulated halos than NFW
at the expense of an extra parameter

Merritt et al (2006):

$$\rho(r) = \rho_e \, e^{-d_n\left[\left(\frac{r}{r_e}\right)^{1/n} - 1\right]}$$

If that looks familiar, it is because the Einasto profile is the 3D equivalent of the Sersic profile used for fittin projected surface densities

ρ_e = density at radius r_e
that contains $\frac{1}{2}$ of the total mass

d_n is a hassle to obtain, but is well approximated by

$$d_n \simeq 3n - \frac{1}{3} + \frac{0.0079}{n} \qquad \text{for } n > \frac{1}{2}$$

Simulated halos have $4.6 < n < 8.2$

In the notation of Navarro et al. (2004), $\alpha = \frac{1}{n}$
who found the mid-point $\alpha \simeq 0.17$

The variable n is sometimes invoked to explain cores in real galaxies, but this is not correct.
Both the change in slope and its location (at very small radii) fail to explain observations.
It is too small an effect - really just a tweak to NFW

Einasto halos do have the nice property of a finite total mass

$$M(r) = 4\pi n r_e^3 \rho_e e^{d_n} d_n^{-3n} \gamma(3n, x)$$

where γ is the incomplete gamma fcn

$x = d_n \left(\frac{r}{r_e}\right)^{1/n}$

integrating $M(r)$ to ∞ turns γ into Γ

$$M_{tot} = 4\pi n r_e^3 \rho_e e^{d_n} d_n^{-3n} \Gamma(3n)$$

Empirical DM Halo — McGaugh et al (2007), Walker et al (2009)

Just fit the portion of the data attributable to dark matter. Adjust M_*/L & choose that which minimizes scatter - in both BTFR and $V_{DM}(R)$, as it happens. Get

$$\log_{10} V(R) = 1.47^{+0.15}_{-0.19} + \frac{1}{2} \log R$$

V in km s^{-1}
R in kpc

slope $\frac{1}{2}$ fixed to NFW value (best fit 0.49)

this can also be expressed in terms of the enclosed mass

$$M_{DM}(R) = 200^{+200}_{-120} \left(\frac{R}{pc}\right)^2$$

M in $M_\odot$
R in pc this time

note: this is not the same as M_{total}, which increases linearly

or

$$g_{DM} = 3^{+3}_{-2} \times 10^{-11} \text{ m s}^{-2}$$