Potential – Density Pairs

<u>Kuzmin disk</u> special case of Toomre' Model (#1)

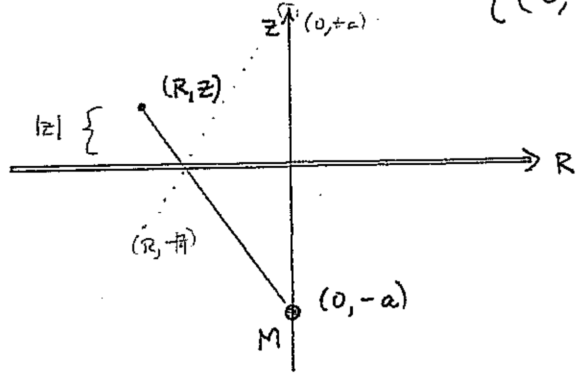
now consider axis-symmetric (cylindrical, not spherical) potential

R now in x-y plane

$$\Phi_K(R,z) = -\frac{GM}{\sqrt{R^2 + (a+|z|)^2}}$$

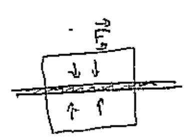
Note that outside the plane, Φ_K is identical to that of a point mass located at

$$(R,z) = \begin{cases} (0,a) & \text{for } z<0 \\ (0,-a) & z>0 \end{cases}$$



Hence, $\nabla^2 \Phi_K = 0$ for $z \neq 0$

Apply Gauss's thm to plane to get



$\oint \nabla\Phi \cdot dI = 4\pi GM$

$\nabla\Phi = 2\pi G\Sigma$

for $z \to 0$

$\Sigma = \frac{1}{2\pi G} \frac{\partial\Phi}{\partial z}\Big|_{z=0}$

$$\Sigma_K(R) = \frac{aM}{2\pi(R^2+a^2)^{3/2}}$$

This is an infintesimally thin disk

$\Sigma_K(z) = \delta(z)$

not physical, but often a good approximation (disk systems are, by their nature, thin)

see BT 2.3.1

Miyamoto & Nagai Φ-ρ pair

potential: $$\Phi_M(R,z) = -\frac{GM}{\sqrt{R^2+(a+\sqrt{z^2+b^2})^2}}$$

combines Plummer's spherical
and Kuzmin's flattened potentials:

$a = 0 \Rightarrow$ Plummer
$b = 0 \Rightarrow$ Kuzmin

from $\nabla^2 \Phi_M$, get complicated expression for ρ_M

(Binney & Tremaine eqn 2-50b)

show transparency, Fig. 2-6 of B&T

M-N density distribution for $b/a = 0.2, 1, \& 10$ ← nearly spherical

very flattened

Bears some passing resemblance to real galaxies

BUT $\rho(R) \sim R^{-3}$ at large arc

real galaxies DISK $\Sigma(R) \sim e^{-R}$ for stars
$\Sigma(R) \sim R^{-1}$ for gas   } often with hard edges

Toomre's models n

Generalize Kuzmin disk by constructing series of linear combinations of Φ-ρ pairs; e.g., by $\frac{\partial^{n-1}\Phi_K}{\partial a^2}$ with $a^2 = 2nR_T^2$

Doing this gives

$$\Sigma_K^n(R) = \Sigma_0\left(1 + \frac{R^2}{2nR_T^2}\right)^{-(n+\frac{1}{2})}$$

in the limit $n \to \infty$, becomes Gaussian

$$\Sigma_K^\infty(R) = \Sigma_0\, e^{-\frac{R^2}{2R_T^2}}$$

. . .

. . . Can play similar generalization game with Miyamoto-Nagai potential (eqns. 2-53 B&T)

Potential of Exponential Disk

The disk components of spiral galaxies (i.e., THE STARS) are generally exponential in nature:

$$\Sigma(R) = \Sigma_0 \, e^{-R/R_d}$$

integrated mass distribution

$$M(R) = 2\pi \int R\,\Sigma(R)\,dR = 2\pi \Sigma_0 R_d^2 \left[1 - e^{-R/R_d}\left(1+\frac{R}{R_d}\right)\right]$$

Hankel transform (like Fourier transform, but for cylindrical things)

$$S(k) = -2\pi G \int_0^\infty J_0(kR)\,\Sigma(R)\,R\,dR$$

J_0 — cylindrical Bessel fn of order zero

For exponential disk

$$S(k) = -\frac{2\pi G \Sigma_0 R_d^2}{\left[1+(kR_d)^2\right]^{3/2}}$$

then

$$\Phi(R,z) = -2\pi G \, \Sigma_0 R_d^2 \int_0^\infty \frac{J_0(kR)\, e^{-k|z|}}{\left[1+(kR_d)^2\right]^{3/2}}\, dk$$

disk mass $\underline{M_d = 2\pi \Sigma_0 R_d^2}$

actually doing this (Freeman 1970) — BT 2.6

gives

$$\Phi(R,0) = -\pi G \Sigma_0 R \left[I_0(y)K_1(y) - I_1(y)K_0(y)\right]$$

$$y \equiv \frac{R}{2R_d}$$

which gives the rotation curve

$$V_c^2(R) = R\frac{\partial \Phi}{\partial R} = 4\pi G \Sigma_0 R_d \, y^2 \left[I_0(y)K_0(y) - I_1(y)K_1(y)\right]$$

peaks @ $2.2 R_d$

$I_n(y)$ = modified Bessel fcn of first kind

$K_n(y)$ = " " " " second "

http://www.astro.umd.edu/~ssm/expdisk.bess

NOTE: $V(R)$ from $\Sigma(R)$ "straight forward", in practice perform numerical Hankel transform of real observed $\Sigma(R)$

$\Sigma(R)$ from $V(R)$ $\longrightarrow$ errors blow up in your face

Logarithmic Potentials

Potentials we've discussed so far arise from finite mass distributions, and so have Keplerian rotation behavior $V_c(r) \sim r^{-1/2}$ at large r.

But real spiral galaxies have flat rotation curves! $V_c \to$ const for large r

$$\text{acceleration} = \frac{V_c^2}{R} = -\frac{\partial \Phi}{\partial R} \propto \frac{1}{R}$$

$$\longrightarrow \quad \Phi \propto \ln R \underset{+\,\text{const}}{} \quad !$$

motivates form

$$\Phi_L = \frac{1}{2} V_0^2 \ln\left(R_c^2 + R^2 + \frac{z^2}{q_\Phi^2}\right) \quad + \text{const}$$

R_c and V_0 are constants (size scale & "depth" of potential)

$q_\Phi \leq 1$

Equipotential surfaces are ellipses with axis ratio q_Φ

Density distribution complex but analytic (B&T eqn 2-54b) new BT eq. 2.71c

show transparency (B&T Fig 2-8) showing equidensity contours

Note that formally $\rho < 0$ when $q_\Phi = 0.7$ and $|z| \gtrsim 7R_c$

Weird density distribution due to "disk + halo" mass distributions

BT 2.3.2

Logarithmic potential are often a good approximation for dark matter halos in an empirical sense, but they are NOT what is predicted by structure formation simulations.

Classical Integrals of Motion

conserved quantities: Energy
Angular momentum
Third Integral *

* For many cases of interest (e.g., axisymmetric potentials) there is an effective 3^{rd} integral of motion that can not generally be expressed analytically, but does limit the possible orbit families.

In cylindrical coordinates R, ϕ, z

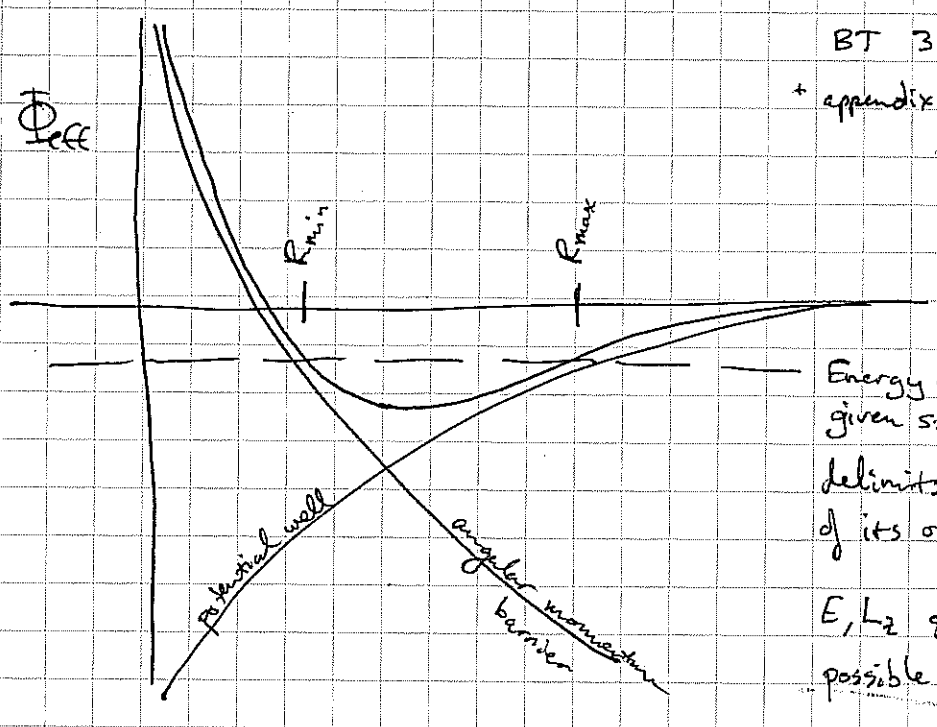
Energy: $$E = \frac{1}{2}\left(V_R^2 + V_\phi^2 + V_z^2\right) + \Phi(R, z)$$ (Hamiltonian representation)

Angular momentum $L_z = R V_\phi = R^2 \Omega = R^2 \dot{\phi}$

Since angular momentum is conserved, can write

effective potential $$\Phi_{eff}(R, z) = \Phi(R, z) + \frac{L_z^2}{2R^2}$$

BT 3.2
+ appendix D4



Energy of a given star delimits range of its orbit

E, L_z specify possible orbits