The Collisionless Boltzmann eqn

BT 4.1

PHASE SPACE

distribution fcn $f(\vec{x}, \vec{v}, t)$ in potential $\Phi(\vec{x}, t)$

of course, Φ generated by mass density in f

so once you know $f(\vec{x}, \vec{v}, t_0)$ you can (in principle) compute any $f(\vec{x}, \vec{v}, t)$

BT define coords $w \equiv (\vec{x}, \vec{v}) = w_1 \cdots w_6$
x, y, z, v_x, v_y, v_z

then $\dot{w} = (\dot{x}, \dot{v}) = (\vec{v}, -\vec{\nabla}\Phi)$

IF mass is conserved (a closed system)
AND there are no collisions causing sudden jumps in f,
f must obey continuity condition:

$$\underbrace{\frac{\partial f}{\partial t}}_{\substack{\text{rate of flow}\\ \text{into volume}}} + \underbrace{\sum_{\alpha=1}^{6} \frac{\partial}{\partial w_\alpha}(f \dot{w}_\alpha)}_{\substack{\text{rate of flow}\\ \text{out of volume}}} = 0$$

(divergence)

using the fact that Φ depends only on $\vec{x}$ and not $\vec{v}$, this simplifies to

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \dot{w}_\alpha \frac{\partial f}{\partial w_\alpha} = 0$$

or in more familiar notation

$$\boxed{\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f - \vec{\nabla}\Phi \cdot \frac{\partial f}{\partial \vec{v}} = 0}$$

Collisionless Boltzmann Equation
also called Vlasov Equation

Seperate species (e.g. dwarfs, giants) must satisfy this

"collisionless" OK if $\quad \eta = \left| \frac{B - D}{f/t_{cross}} \right| \ll 1$

B = birthrate
D = deathrate

> Phase space ~~volume~~ is conserved —
you can mix in empty volume,
but you can't compress f —
you can't just pile more stars into the
same volume in configuration space
w/o also affecting their momenta
(velocity space ~~is~~ part of distribution)

> Limitations when applied to stars
in stellar systems

- finite lifetimes. Stars don't live
forever, so the implicit assumption
of an eternal, constant point mass
must break down at some point.
In practice, OK for M dwarfs ($t_* \gg$ age of U)
but not O stars ($t \ll$ crossing time)
Cutting it closer $M \lesssim 1.5 M_\odot$ OK ($t \sim 1$ Gyr)

- Correlations between stars
In practice, need to consider finite
(not infitesimal) volumes containing
finite number of real stars.
Obvious assumption is ~~$\overline{Nf}$~~
~~where~~ to average over finite volumes to get
f. This assumes stars are uncorrelated.
Probably OK for old, well mixed stars, but not guaranteed

Jeans equations - integral of d.f. $f(x,\dot{x})$ corresponding to conservation of energy, angular moment & 3rd integral

BT ch. 4

f is a fcn of 7 variables, so obtaining soln's to the collisionless Boltzmann eqn challenging in practice "Simplify" by taking moments (integrate over all $\vec{v}$. Note $f \to 0$ for $v \to \infty$) gives Jeans equations:

Note: can integrate again over $\vec{x}$ to obtain tensor virial theorem (B.T. 4. These steps lose information by averaging over f

For an isothermal system, $f(v)$ is Maxwellian e^{-E/σ^2}

$$\boxed{\begin{aligned}
&\frac{\partial \nu}{\partial t} + \frac{\partial(\nu \bar{v}_i)}{\partial x_i} = 0 \\
&\frac{\partial(\nu \bar{v}_j)}{\partial t} + \frac{\partial(\nu \overline{v_i v_j})}{\partial x_i} + \nu \frac{\partial \Phi}{\partial x_j} = 0 \\
&\nu \frac{\partial \bar{v}_j}{\partial t} + \nu \bar{v}_i \frac{\partial \bar{v}_j}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial(\nu \sigma_{ij}^2)}{\partial x_i}
\end{aligned}}$$

where

$$\nu = \int f \, d^3v \qquad \nu = \text{space density of stars}$$

$$\bar{v}_i = \frac{1}{\nu} \int f v_i \, d^3v \qquad \bar{v}_i = \text{mean velocity in } i^{\text{th}} \text{ direction}$$

similarly

$$\overline{v_i v_j} = \frac{1}{\nu} \int v_i v_j f \, d^3v$$

"cross-talk" important only for non-symmetric mass distributions (like barred spirals!)

and $\sigma_{ij}^2 = \overline{v_i v_j} - \bar{v}_i \bar{v}_j$

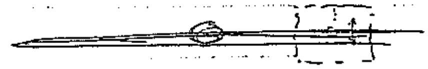
note that

$-\dfrac{\partial(\nu \sigma_{ij}^2)}{\partial x_i}$ is like a pressure force $-\nabla p$

is really a stress tensor that in effect allows for different pressures in different directions. This term is important in quasi-spherical, triaxial systems (e.g. Ellipticals dark matter halos) and so these are often referred to as "pressure supported" systems.

Application of Jeans Equations: BM 10.4.4

Surface mass density in solar neighborhood



- Poisson eqn: $\nabla^2 \Phi = -\vec{\nabla}\cdot\vec{F} = +4\pi G\rho$ $\qquad \rho = \bar{m}\nu$

in cylindrical coords, this becomes

$$\frac{1}{R}\frac{\partial}{\partial R}(R F_R) + \frac{\partial F_z}{\partial z} = -4\pi G \rho$$

$F_R = -\frac{V_c^2}{R}$: $V_c \approx$ const. in vicinity of sun, so $\frac{\partial F_R}{\partial R} \approx 0$

so $\rho \simeq -\frac{1}{4\pi G}\frac{\partial F_z}{\partial z}$

$\Sigma = 2\int_0^\infty \rho\, dz \simeq \frac{-F_z}{2\pi G}$ $\qquad \left(\int_{-\infty}^{\infty} = 2\int_0^\infty\right)$

- Jeans eqn: $\nu F_z = \frac{\partial(\nu\sigma_z^2)}{\partial z} + \frac{1}{R}\frac{\partial}{\partial R}(R\nu\sigma_{Rz}^2)$

make further approximation that R & z separable:

$\Phi(R,z) = \Phi(R) + \Phi(z)$ so $\sigma_{Rz}^2 \approx 0$

so now have F_z to get $\Sigma_\odot$:

$$\Sigma = -\frac{1}{2\pi G\nu}\frac{\partial(\nu\sigma_z^2)}{\partial z}$$

So, need to observe the number density distribution $\nu(z)$ of some population of stars above the plane and its velocity dispersion σ

ν typically modelled as exp: $\nu_0 e^{-z/z_0}$ or $\nu_0 \operatorname{sech}^2(z/z_d)$

Kuijken & Gilmore (1991) find

$\Sigma_\odot(|z|<1.1\,\mathrm{kpc}) = 71 \pm 6\ M_\odot\,\mathrm{pc}^{-2}$ $\qquad$ most uncertainty from validity of assumptions

of which $\Sigma_{d,\odot} = 48 \pm 9\ M_\odot\,\mathrm{pc}^{-2}$ $\qquad \Sigma_* \approx 35\ M_\odot\,\mathrm{pc}^{-2}$

so no evidence for in-disk Oort discrepancy $\qquad \Sigma_g \approx 13\ M_\odot\,\mathrm{pc}^{-2}$

but some for an out-of-disk DM halo.

A few notional time scales

- crossing time $\quad t_c = \frac{2R}{V} \qquad \frac{\text{diameter}}{v \text{ or } \sigma}$

typical time to go from one side of a system to the other

Question: will an O star live many crossing times?

- dynamical time $\quad t_d = \sqrt{\frac{3\pi}{16 G\rho}}$

typical orbital time in homogeneous sphere of density ρ

- relaxation time $\quad \frac{t_R}{t_c} = \frac{N}{48 f^2} \approx \frac{N}{6 \ln(N/2)}$

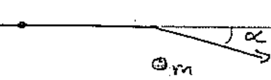
↳ N objects carrying a fraction f of the mass

typical time to forget initial conditions

for MW (smoother) w/ $N \approx 10^{11}$, $t_R \gg$ age of Universe

for GC (grainier) w/ $N \approx 10^6$, $t_R \approx 10$ Gyr ~ age of Universe

Relaxation is the cumulative result of many weak gravitational encounters that only nudge the vector $\alpha = \frac{\Delta v_\perp}{v} = \frac{2Gm}{bv^2}$



Strong encounters (w/ ΔP.E. ~ K.E.) are rare in the solar neighborhood $\quad t_{strong} \sim 10^{15}$ yr $\gg$ U

Galactic Structure & Kinematics

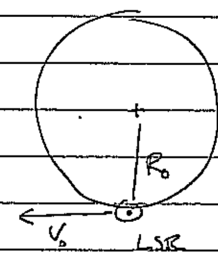
How to find

R_0:	V_0:
- Centroid of Globular Clusters	- solar motion wrt G.C.s (but do GCs rotate?)
- " " Halo tracers (e.g. RR Lyrae)	- Proper motion of Sgr A* (need R_0!)
- kinematic $v_r = 0$ along $d = 2 R_0 \cos(2l)$	- local escape velocity -
- expanding masers @ G.C. $R_0 = 7.9 \pm 0.8$ kpc	$V_{esc} \simeq \sqrt{2}\, V_c$ ($\sqrt{2}$ depends on details of the mass model)

Sharp cut-off in # stars with $v > V_c + 63$ km/s

Galactic Constants

V_0, R_0



R_0 = distance to Galactic Center
≈ 8 kpc ± 0.5 all systematics
7.9 - 8.3 recently

V_0 = orbital velocity of LSR → also called Θ_0
≈ 230 km s^{-1} ± 20 all systematics
Official IAU values 220, 8.5 for many years

angular velocity of LSR now pretty well known from proper motion of Sgr A*

$$\Omega_0 = \frac{V_0}{R_0} \approx 30 \text{ km s}^{-1} \text{ kpc}^{-1} \pm 0.5 \text{ or so}$$

measured with VLBA long after IAU values were set

Oort constants

A local shear (differential rotation)

B vorticity (spin or circulation of local fluid element)

$A \simeq 15$ km s^{-1} kpc ; $B \simeq -13$ km s^{-1} kpc^{-1}

Also C & D $\simeq 3$ km s^{-1} kpc^{-1} for non-axis-symmetric potentials e.g. the Galactic Bar

K epicyclic frequency

$K \approx 37$ km s^{-1} kpc^{-1}