Chapter 7

Orbits, Integrals, and Chaos

In n space dimensions, some orbits can be formally decomposed into n independent periodic motions. These are the *regular* orbits; they may be represented as winding paths on an n-dimensional torus. On the other hand, *irregular* or stochastic orbits defy any such representation; in theory such orbits may wander anywhere permitted by conservation of energy.

7.1 Constants & Integrals of Motion

Constants of motion are functions of phase-space coordinates *and* time which are constant along orbits:

$$C(\mathbf{r}(t), \mathbf{v}(t), t) = \text{const.}, \tag{7.1}$$

where $\mathbf{r}(t)$ and $\mathbf{v}(t) = d\mathbf{r}/dt$ are a solution to the equations of motion. The function $C(\mathbf{r}, \mathbf{v}, t)$ must be constant along *every* orbit, with a value which depends on the orbit. In a phase-space of $2n$ dimensions there are always $2n$ independent constants of motion. For example, the $2n$ initial conditions $(\mathbf{r}_0, \mathbf{v}_0)$ of an orbit are constants of motion; given phase-space coordinates $(\mathbf{r}, \mathbf{v})$ at time t, integrate the orbit backwards to $t = 0$ and read off the initial $(\mathbf{r}_0, \mathbf{v}_0)$.

Integrals of motion are functions of phase-space coordinates alone which are constant along orbits:

$$I(\mathbf{r}(t), \mathbf{v}(t)) = \text{const.}. \tag{7.2}$$

An integral of motion can't depend on time; thus all integrals are constants of motion, but not all constants are integrals. Integrals come in two varieties: *isolating* and *non-isolating*. Isolating integrals are important because they constrain the shapes of orbits; in a phase-space of $2n$ dimensions, an isolating integral defines a hypersurface of $2n - 1$ dimensions. *Regular orbits* are those which have $N = n$ isolating integrals; in such cases each orbit is confined to a hypersurface of $2n - N = n$ dimensions.

7.2 Orbits in Spherical Potentials

Consider the motion of a star in a spherically-symmetric potential, $\Phi = \Phi(|\mathbf{r}|)$. The orbit of the star remains in a plane perpendicular to the angular momentum vector, so it's natural to adopt a polar coordinate system; call the coordinates $R = |\mathbf{r}|$ and ϕ. The system has $n = 2$ degrees of freedom, so the phase space has 4 dimensions. All orbits in spherical potentials are regular; they have two isolating integrals.

The **equations of motion** can be derived by starting with the lagrangian,

$$L(R,\phi,\dot{R},\dot{\phi}) = \frac{1}{2}(\dot{R}^2 + R^2\dot{\phi}^2) - \Phi(R)\,, \tag{7.3}$$

where $\dot{R} = dR/dt$ and $\dot{\phi} = d\phi/dt$. Differentiating with respect to $\dot{R}$ and $\dot{\phi}$ yields the momenta conjugate to R and ϕ,

$$\frac{\partial L}{\partial \dot{R}} = \dot{R} = v_{\mathrm{R}}\,, \qquad \frac{\partial L}{\partial \dot{\phi}} = R^2\dot{\phi} = Rv_\phi = J\,; \tag{7.4}$$

here v_{R} and v_ϕ are velocities in the radial and azimuthal directions. The hamiltonian may now be expressed as a function of the coordinates and conjugate momenta:

$$H(R,\phi,v_{\mathrm{R}},J) = \frac{1}{2}(v_{\mathrm{R}}^2 + J^2/R^2) + \Phi(R)\,. \tag{7.5}$$

Then the equations of motion are

$$\begin{aligned} \frac{dR}{dt} &= \frac{\partial H}{\partial v_{\mathrm{R}}} = v_{\mathrm{R}}\,, & \frac{dv_{\mathrm{R}}}{dt} &= -\frac{\partial H}{\partial R} = -\frac{d\Phi}{dR} + \frac{J^2}{R^3}\,, \\ \frac{d\phi}{dt} &= \frac{\partial H}{\partial J} = \frac{J}{R^2}\,, & \frac{dJ}{dt} &= -\frac{\partial H}{\partial \phi} = 0\,. \end{aligned} \tag{7.6}$$

Here $dJ/dt = 0$ because (7.5) is independent of the conjugate coordinate ϕ.

The two independent integrals of motion are thus the total energy E, numerically equal to the value of H, and the angular momentum J. These quantities are given by

$$E = \frac{1}{2}(v_{\mathrm{R}}^2 + v_\phi^2) + \Phi(R)\,, \qquad J = Rv_\phi\,. \tag{7.7}$$

Each of these integrals defines a hypersurface in phase space, and the orbit is confined to the intersection of these hypersurfaces. This can be visualized by ignoring the ϕ coordinate and drawing surfaces of constant E and J in the three-dimensional space $(R, v_{\mathrm{R}}, v_\phi)$, as in Fig. 7.1. Surfaces of constant E are figures of revolution about the R axis, while surfaces of constant J are hyperbolas in the (R, v_ϕ) plane. The intersection of these surfaces is a closed curve, and an orbit travels around this curve.

For an orbit of a given J, the system may be reduced to one degree of freedom by defining the *effective potential*,

$$\Psi(R) = \Phi(R) + \frac{J^2}{2R^2} \tag{7.8}$$

the corresponding equations of motion are then just

$$\frac{dR}{dt} = v_{\mathrm{R}}\,, \qquad \frac{dv_{\mathrm{R}}}{dt} = -\frac{d\Psi}{dR}\,. \tag{7.9}$$

Because $\Psi(R)$ diverges as $R \to 0$, the star is energetically prohibited from coming too close to the origin, and shuttles back and forth between turning points R_{min} and R_{max}.

In addition to its periodic radial motion described by (7.9), a star also executes a periodic azimuthal motion as it orbits the center of the potential. If the radial and azimuthal periods are incommensurate, as is usually the case, the resulting orbit never returns to its starting point in phase space;

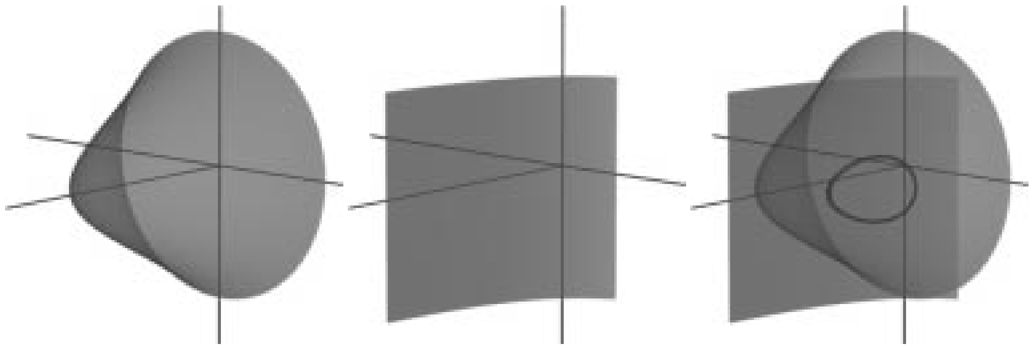

Figure 7.1: An orbit in a Hernquist (1993) potential as the intersection of surfaces of constant E and J. Left: surface of constant E. Middle: surface of constant J. Right: intersection of these surfaces.

in coordinate space such an orbit is a rosette (BT87, Fig. 3-1). The Keplerian potential is a very special case in which the radial and azimuthal periods of all bound orbits are *equal*. The only other potential in which all orbits are closed is the harmonic potential generated by a uniform sphere; here the radial period is half the azimuthal one and all bound orbits are ellipses centered on the bottom of the potential well. Thus in the Keplerian case all stars advance in azimuth by $\Delta\phi = 2\pi$ between successive pericenters, while in the harmonic case they advance by $\Delta\phi = \pi$. Galaxies typically have mass distributions intermediate between these extreme cases, so most orbits in spherical galaxies are rosettes advancing by $\pi < \Delta\phi < 2\pi$ between pericenters.

This combination of radial and azimuthal motions can be represented as a path on a 2-torus; that is, on a rectangle made periodic by gluing both opposing pairs of edges together. Associate the long direction on the torus with the azimuthal direction of the orbit, and the short direction on the torus with the closed curve (7.9) produces on the $(R, v_{\rm R})$ plane; the result is known as an *invariant torus*. An orbit winds solenoid-like on the surface of the torus, typically making between one and two turns through the hole of the torus for each turn made around its circumference. Points on the surface of the torus may be parameterized by a pair of angles (θ_1, θ_2); moreover, by stretching the torus appropriately, the motion of a star can be described by a pair of linear relations:

$$\theta_1(t) = \theta_1(0) + \omega_1 t\,, \qquad \theta_2(t) = \theta_2(0) + \omega_2 t\,, \tag{7.10}$$

where the ω_i are integrals of motion. Together, (θ_1, θ_2) and their conjugate angular frequencies or *actions* (ω_1, ω_2) define a coordinate system for the four-dimensional phase space in which the hamiltonian takes the simplest possible form,

$$H(\theta_1, \theta_2, \omega_1, \omega_2) = \frac{1}{2}(\omega_1^2 + \omega_2^2)\,; \tag{7.11}$$

these coordinates are known as *action-angle variables*.

7.3 Orbits in Axisymmetric Potentials

In describing axisymmetric galaxy models it's natural to use cylindrical coordinates (R, ϕ, z), where R and ϕ are polar coordinates in the equatorial plane, and z is the coordinate perpendicular to that

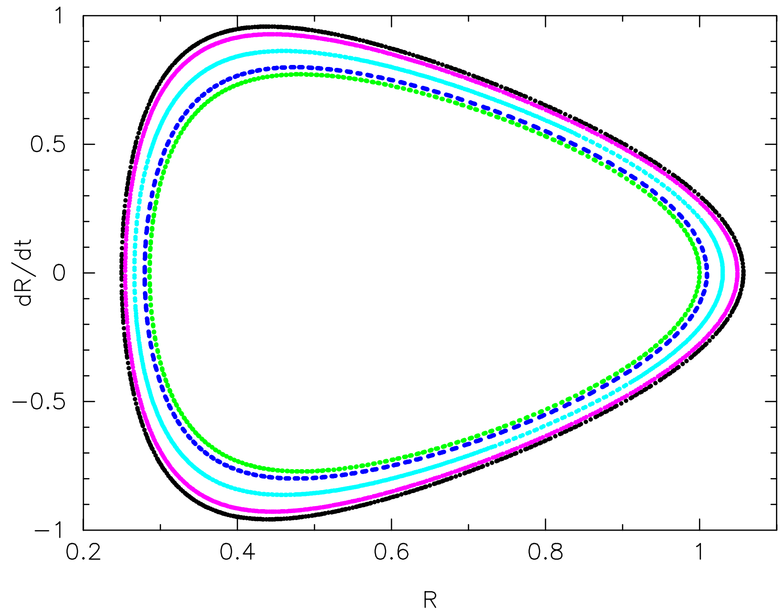

Figure 7.2: Surface of section for five orbits in the logarithmic potential.

plane. In these coordinates, the potential has the form $\Phi = \Phi(R,z)$. The equations of motion are identical to (7.6), with additional expressions for z and v_z:

$$\frac{dz}{dt} = v_z\,, \qquad \frac{dv_z}{dt} = -\frac{\partial\Phi}{\partial z}\,. \tag{7.12}$$

Once again, there are two classical integrals of motion:

$$E = \frac{1}{2}(v_R^2 + v_\phi^2 + v_z^2) + \Phi(R,z)\,, \qquad J_z = R v_\phi\,. \tag{7.13}$$

Just as for spherical potentials, it's possible to define an effective potential

$$\Psi(R,z) = \Phi(R,z) + \frac{J_z^2}{2R^2} \tag{7.14}$$

Instead of governing motion along a line as in the spherical case, the effective potential now governs the star's motion in the *meridional plane*, which rotates about the z axis with angular velocity $\omega = J_z/R^2$. The radial motion is described by (7.9), while the vertical motion is described by (7.12) with Φ replaced by Ψ.

On the meridional plane, which has coordinates (R,z), the effective potential Ψ has a minimum at $R > 0$ and $z = 0$ and a steep angular momentum barrier as $R \to 0$ (BT87, Fig. 3-2). If only the energy E constrains the motion of a star on this plane, one might expect a star to travel everywhere within some closed contour of constant Ψ. But in many cases this is *not* observed; instead, stars launched from rest at different points along a contour of constant Ψ follow distinct trajectories. This implies the existence of a *third integral* besides the classic integrals given by (7.13). No general expression for a third integral exists.

The *surface of section* is a simple and elegant technique for visualizing the non-classical integrals of an orbit. It is normally only applicable to systems with $n = 2$ dimensions, but in the axisymmetric case, we can use a surface of section to analyze orbits in the meridional plane. To construct a surface of section for this case, simply follow the orbit and plot the phase-space coordinates $(R, \dot{R})$ whenever the orbit crosses the $z = 0$ plane in an upward direction ($\dot{z} > 0$). Fig. 7.2 presents results for five different orbits in the axisymmetric logarithmic potential,

$$\Phi(R, z) = \frac{1}{2} \ln(R_c^2 + R^2 + z^2/q^2), \tag{7.15}$$

where R_c is a core radius and $q < 0$ is the axial ratio. All of these orbits started from $(R, \phi, z) = (1, 0, 0)$ with angular momentum $J_z = 0.4$, and all have the same total energy E. If E and J_z were the only integrals of motion, we would expect each orbit to cross the surface of section *anywhere* within some energetically permitted region. Instead, each orbit defines a distinct contour on the $(R, \dot{R})$ plane; these contours are level surfaces of the mysterious third integral.

The existence of a third integral implies that a star's orbit is a combination of *three* periodic motions: radial, azimuthal, and vertical. Thus the orbit can be represented as a path on an invariant 3-torus, with action-angle variables (θ_i, ω_i), where $i = 1, 2, 3$.

Some axisymmetric potentials do have orbits which wander everywhere energetically permitted on the meridional plane. For such orbits, the description in terms of motion on a invariant 3-torus breaks down. These are examples of *irregular* or *stochastic orbits* in an axisymmetric potential; they respect only the two classical integrals, E and J_z.

7.4 Orbits in Non-Axisymmetric Potentials

Non-axisymmetric potentials, with $\Phi = \Phi(x, y)$ or $\Phi(x, y, z)$ in Cartesian coordinates (x, y, z), admit an even richer variety of orbits. The only classical integral of motion in such a potential is the energy,

$$E = \frac{1}{2}|\mathbf{v}|^2 + \Phi(\mathbf{r}). \tag{7.16}$$

Some potentials nonetheless permit other integrals of motion, and in such potentials regular orbits may be mapped onto invariant tori. But not all regular orbits can be continuously deformed into one another; consequently, orbits can be grouped into topologically distinct *orbit families*. Each regular orbit family will generally require a different set of invariant tori. Some sense of the variety of possible orbits in non-axisymmetric galaxies is available by examining orbits in separable and scale-free potentials.

7.4.1 Separable Potentials

In a *separable potential* all orbits are regular and the mapping to the invariant tori can be constructed analytically; all integrals of motion are known. Separable potentials are rather special, mathematically speaking, and it's highly unlikely that real galaxies have such potentials. However, numerical experiments show that non-axisymmetric galaxy models with finite cores or shallow cusps usually generate potentials with many key features of separable potentials.

The orbits in a separable potential may be classified into distinct *families*, each associated with a set of closed and stable orbits. In two dimensions, for example, there are two types of closed, stable orbits; one type (i) oscillates back and forth along the major axis, and the other type (ii) loops around the center. Because these orbits are stable, other orbits which start nearby will remain nearby at later times. The families associated with types (i) and (ii) are known as *box* and *loop*

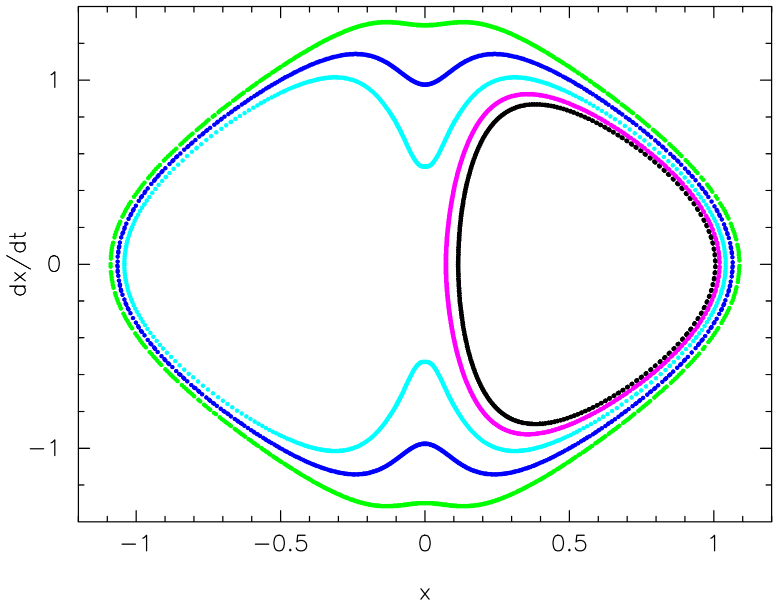

Figure 7.3: Surface of section for five orbits in the logarithmic potential.

orbits, respectively (BT87, Ch. 3.3.1). Fig 7.3 presents a surface of section for orbits in the non-axisymmetric logarithmic potential

$$\Phi(x,y) = \frac{1}{2}\ln(R_c^2 + x^2 + y^2/q^2)\,. \tag{7.17}$$

(Technically speaking, this is *not* a separable potential, but as noted above it shares many features with separable potentials for $R_c > 0$.) This surface of section was generated by following each orbit and plotting $(x,\dot{x})$ whenever the orbit crossed the $y = 0$ axis with $\dot{y} > 0$. All five orbits shown have the same energy; the outer three are box orbits, while the inner two are loops.

In three dimensions, a separable potential permits four distinct orbit families:

1. *box* orbits,
2. *short-axis tube* orbits,
3. *inner long-axis tube* orbits, and
4. *outer long-axis tube* orbits.

The short-axis tubes are orbits which loop around the short (minor) axis, while long-axis tubes loop around the long (major) axis. The two families of long-axis tube orbits arise from different closed stable orbits and explore different regions of space (BT87, Fig. 3-20). No 'intermediate-axis tube' orbits exist since closed orbits looping around the intermediate axis are unstable. In general, triaxial potentials with cores have orbit families much like those in separable potentials.

7.4.2 Scale-Free Potentials

In scale-free models all properties have either a power-law or a logarithmic dependence on radius. In particular, scale-free models with density profiles proportional to r^{-2} have logarithmic potentials

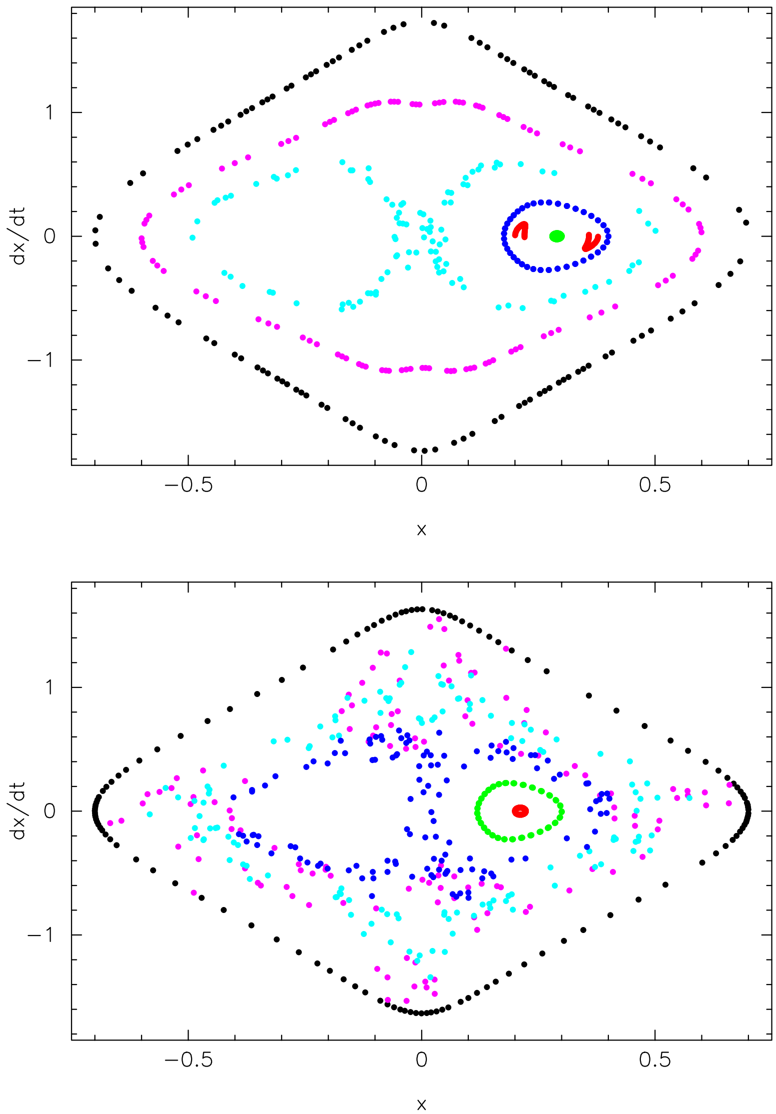

Figure 7.4: Surface of section for five orbits in the perturbed logarithmic potential.

and flat rotation curves. While real galaxies are not entirely scale-free, such steep power-law density distributions are reasonable approximations to the central regions of some elliptical galaxies and to the halos of galaxies in general.

If the density falls as r^{-2} or faster, then box orbits are replaced by minor orbital families called *boxlets* (Gerhard & Binney 1985, Miralda-Escude & Schwarzschild 1989). Each boxlet family is associated with a closed and stable orbit arising from a resonance between the motions in the x and y directions.

7.5 Irregular orbits

Fig. 7.4 presents surfaces of section for orbits in the potential

$$\Phi(x,y) = \frac{1}{2}\ln(R_c^2 + x^2 + y^2/q^2 - \sqrt{x^2+y^2}(x^2-y^2)/R_e)\,, \qquad (7.18)$$

where R_e is a second scale radius (*not* to be confused with the effective radius of a de Vaucouleurs profile!). In the limit $R_e \to \infty$ this reduces to (7.17). The top panel shows results for $R_e = 3.0$, while the lower panel shows $R_e = 1.5$; note the larger chaotic zone in the latter case.

In principle, an irregular orbit can wander everywhere on the phase-space hypersurface of constant E, but in actuality such orbits show a complicated and often fractal-like structure.