The Collisionless Boltzmann eqn

BT 4.1

PHASE SPACE

distribution fcn $f(\vec{x}, \vec{v}, t)$ in potential $\Phi(\vec{x}, t)$

of course, Φ generated by mass density in f.

so ~~if~~ once you know $f(\vec{x}, \vec{v}, t_0)$ you can

(in principle) compute any $f(\vec{x}, \vec{v}, t)$

BT define coords $w \equiv (\vec{x}, \vec{v}) = w_1 \cdots w_6$

x, y, z, v_x, v_y, v_z

then $\dot{w} = (\dot{x}, \dot{v}) = (\vec{v}, -\vec{\nabla}\Phi)$

IF mass is conserved (a closed system)

AND there are no collisions causing sudden jumps in f,

f must obey continuity condition:

$$\underbrace{\frac{\partial f}{\partial t}}_{\substack{\text{rate of flow} \\ \text{into volume}}} + \underbrace{\sum_{\alpha=1}^{6} \frac{\partial}{\partial w_\alpha}(f \dot{w}_\alpha)}_{\substack{\text{rate of flow} \\ \text{out of volume}}} = 0$$

(divergence)

using the fact that Φ depends only on $\vec{x}$ and not $\vec{v}$,

this simplifies to

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \dot{w}_\alpha \frac{\partial f}{\partial w_\alpha} = 0$$

or in more familiar notation

$$\boxed{\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f - \vec{\nabla}\Phi \cdot \frac{\partial f}{\partial \vec{v}} = 0}$$

Collisionless Boltzmann Equation

also called Vlasov Equation

~~also~~ Seperate species (e.g. dwarfs, giants) must satisfy this

"collisionless" OK if $\eta = \left| \frac{B - D}{f/t_{cross}} \right| \ll 1$

B = birthrate

D = deathrate

› Phase space ~~volume~~ is conserved –
you can mix in empty volume,
but you can't compress f –
you can't just pile more stars into the
same volume in configuration space
w/o also affecting their momenta
(velocity space ~~is~~ part of distribution)

› Limitations when applied to stars
in stellar systems

– finite lifetimes. Stars don't live
forever, so the implicit assumption
of an eternal, constant point mass
must break down at some point.
In practice, OK for M dwarfs ($t_* \gg$ age of U)
but not O stars ($t \ll$ crossing time)
Cutting it closer $M \lesssim 1.5 M_\odot$ OK ($t \sim 1$ Gyr)

– Correlations between stars
In practice, need to consider finite
(not infitesimal) volumes containing
finite number of real stars.
Obvious assumption is ~~$f = Nf$~~
~~where $\bar{f}$~~ to average over finite volumes to get
$\bar{f}$. This assumes stars are uncorrelated.
Probably OK for old, well mixed stars, but not guaranteed

Jeans equations - integral of d.f. $f(x, \dot{x})$ corresponding to conservation of energy, angular moment & 3rd integral

BT ch. 4

f is a fcn of 7 variables, so obtaining sol'ns to the collisionless Boltzmann eqn challenging in practice

"Simplify" by taking moments (integrate over all $\vec{v}$. Note $f \to 0$ for $v \to \infty$

gives

Note: can integrate again over $\vec{x}$ to obtain tensor virial theorem (B.T. 4. These steps lose information by averaging over f

For an isothermal system, $f(v)$ is Maxwellian e^{-E/σ^2}

Jeans equations:

$$\boxed{\begin{aligned} &\frac{\partial \nu}{\partial t} + \frac{\partial(\nu \bar{v}_i)}{\partial x_i} = 0 \\ &\frac{\partial(\nu \bar{v}_j)}{\partial t} + \frac{\partial(\nu \overline{v_i v_j})}{\partial x_i} + \nu \frac{\partial \Phi}{\partial x_j} = 0 \\ &\nu \frac{\partial \bar{v}_j}{\partial t} + \nu \bar{v}_i \frac{\partial \bar{v}_j}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial(\nu \sigma_{ij}^2)}{\partial x_i} \end{aligned}}$$

where

$$\nu = \int f \, d^3v \qquad \nu = \text{space density of stars}$$

$$\bar{v}_i = \frac{1}{\nu} \int f v_i \, d^3v \qquad \bar{v}_i = \text{mean velocity in } i^{\text{th}} \text{ direction}$$

similarly

$$\overline{v_i v_j} = \frac{1}{\nu} \int v_i v_j f \, d^3v$$

and

$$\sigma_{ij}^2 = \overline{v_i v_j} - \bar{v}_i \bar{v}_j$$

"cross-talk" important only for non-symmetric mass distributions (like barred spirals!)

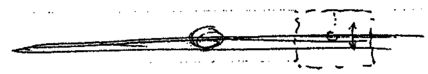
note that

$$-\frac{\partial(\nu \sigma_{ij}^2)}{\partial x_i} \text{ is like a pressure force } -\nabla p$$

is really a stress tensor that in effect allows for different pressures in different directions. This term is important in quasi-spherical, triaxial systems (e.g. Ellipticals, dark matter halos) and so these are often referred to as "pressure supported" systems.

Application of Jeans Equations: BM 10.4.4

Surface mass density in solar neighborhood



- Poisson eqn: $\nabla^2 \Phi = -\vec{\nabla}\cdot\vec{F} = +4\pi G\rho$ $\qquad \rho = \bar{m}\nu$

in cylindrical coords, this becomes

$$\frac{1}{R}\frac{\partial}{\partial R}(RF_R) + \frac{\partial F_z}{\partial z} = -4\pi G\rho$$

$$F_R = -\frac{V_c^2}{R} \; : \quad V_c \approx \text{const. in vicinity of sun, so } \frac{\partial F_R}{\partial R} \approx 0$$

so $$\rho \simeq -\frac{1}{4\pi G}\frac{\partial F_z}{\partial z}$$

$$\Sigma = 2\int_0^\infty \rho\, dz \simeq \frac{-F_z}{2\pi G} \qquad \left(\int_{-\infty}^{\infty} = 2\int_0^\infty\right)$$

- Jeans eqn: $$\nu F_z = \frac{\partial(\nu\sigma_z^2)}{\partial z} + \frac{1}{R}\frac{\partial}{\partial R}(R\nu\sigma_{Rz}^2)$$

make further approximation that R & z seperable:

$$\Phi(R,z) = \Phi(R) + \Phi(z) \quad \text{so} \quad \sigma_{Rz}^2 \approx 0$$

so now know F_z to get $\Sigma_\odot$:

$$\Sigma = -\frac{1}{2\pi G\nu}\frac{\partial(\nu\sigma_z^2)}{\partial z}$$

So, need to observe the number density distribution $\nu(z)$ of some population of stars above the plane and its velocity dispersion σ

ν typically modelled as exp: $\nu_0 e^{-z/z_0}$ or $\nu_0 \operatorname{sech}^2(z/z_0)$

Kuijken & Gilmore (1991) find

$\Sigma_\odot(|z|<1.1\text{ kpc}) = 71 \pm 6\ M_\odot\,pc^{-2}$ most uncertainty from validity of assumptions

of which $\Sigma_{d,\odot} = 48 \pm 9\ M_\odot\,pc^{-2}$ $\qquad \Sigma_* \approx 35\ M_\odot\,pc^{-2}$

so no evidence for in-disk Oort discrepancy $\qquad \Sigma_g \approx 13\ M_\odot\,pc^{-2}$

but some for an out-of-disk DM halo.