Phase space $f(\vec{x}, \vec{v}, t)$: the distribution function

describes the density distribution of particles in both configuration space (x, y, z) and momentum (v_x, v_y, v_z).

For a stationary (stable) system of widely separated stars, we have the collisionless Boltzmann eqn

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f - \vec{\nabla} \Phi \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

Note that the density in configuration space ν is just the integral over f in momentum: $\nu = \int f d^3v$

We usually use this to break the collisionless Boltzmann eqn into the

Jean's eqns

$$\frac{\partial \nu}{\partial t} + \frac{\partial (\nu \bar{v}_i)}{\partial x_i} = 0$$

$$\frac{\partial (\nu \bar{v}_j)}{\partial t} + \frac{\partial (\nu \overline{v_i v_j})}{\partial x_i} + \nu \frac{\partial \Phi}{\partial x_j} = 0$$

$$\nu \frac{\partial \bar{v}_j}{\partial t} + \nu \bar{v}_i \frac{\partial \bar{v}_j}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial (\nu \sigma_{ij}^2)}{\partial x_i}$$

where we have the "moments" of f

$$\nu = \int f d^3v \qquad \overline{v_i v_j} = \frac{1}{\nu} \int v_i v_j f d^3v$$

$$\bar{v}_i = \frac{1}{\nu} \int v_i f d^3v \qquad \sigma_{ij}^2 = \overline{v_i v_j} - \bar{v}_i \bar{v}_j$$

> Phase space ~~volume~~ is conserved —
you can mix in empty volume,
but you can't compress f —
you can't just pile more stars into the
same volume in configuration space
w/o also affecting their momenta
(velocity space ~~is~~ part of distribution)

> Limitations when applied to stars
in stellar systems

- finite lifetimes. Stars don't live
forever, so the implicit assumption
of an eternal, constant point mass
must break down at some point.
In practice, OK for M dwarfs ($t \gg$ age of u)
but not O stars ($t \ll$ crossing time)
Cutting it closer $M \lesssim 1.5 M_\odot$ OK ($t \sim 1$ Gyr)

- Correlations between stars
In practice, need to consider finite
(not infitesimal) volumes containing
finite number of real stars.
Obvious assumption is ~~$\rho = Nf$~~
~~where $\bar{f}$~~ to average over finite volumes to get
$\bar{f}$. This assumes stars are uncorrelated.
Probably OK for old, well mixed stars, but not guaranteed

The Jean's eqn for obtaining the circular velocity of a test particle from the moments of a population of tracers on non-circular orbits:

$$V_c^2(R) = \langle V_\phi^2 \rangle + \langle V_R^2 \rangle \left(1 + \frac{\partial \ln \nu}{\partial \ln R} + \frac{\partial \ln \langle V_R^2 \rangle}{\partial \ln R} \right)$$

where

V_c is the circular velocity of the gravitational potential

V_ϕ is the [quasi-circular] velocity in the tangential direction

V_R is the velocity in the radial direction

ν is the density of tracers (e.g. stars)

so $\frac{\partial \ln \nu}{\partial \ln R}$ is the slope of the radial variation of $\nu(R)$

$\frac{\partial \ln \langle V_R^2 \rangle}{\partial \ln R}$ is the slope of the radial variation of the radial velocity dispersion

$\langle V_\phi^2 \rangle$ & $\langle V_R^2 \rangle$ are the averages over the population, i.e., the net rotation (V_ϕ) and the radial velocity dispersion $\langle V_R^2 \rangle^{1/2} = \sigma_r$

example application: Gaia data for Milky Way
Eilers et al 2019
McGaugh 2019

Cosmological Framework

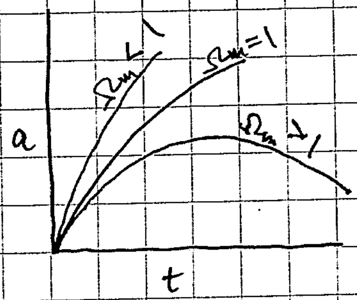
Dark matter halos are thought to form by gravitational collapse of over-dense regions in an otherwise expanding universe.

$a = \frac{1}{1+z}$

A little necessary context:

Friedmann eqn: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$

scale size — mass density — curvature — cosmological constant



a is the scale size of the universe — a dimensionless quantity that encodes the physical separation between comoving coordinate tracers (e.g., galaxies w/ zero peculiar motion)

a(t) is the expansion history of the universe following from the solution of Friedmann's eqn.

Note that the Hubble parameter $H = \frac{\dot{a}}{a}$ is the expansion rate.

H must vary with time; it's current measured value is the misnamed Hubble "constant" $H_0 = \left(\frac{\dot{a}}{a}\right)_0$ measured now at $t = t_0$, the age of the U.

It is also convenient to define the density parameter $\Omega_m = \frac{\rho}{\rho_{crit}}$

which is the ratio of the actual mass density to the critical density

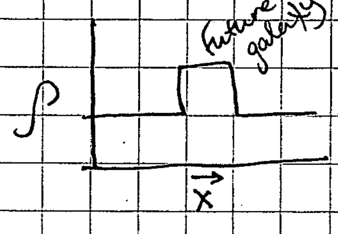
$\rho_{crit} = \frac{3H^2}{8\pi G}$ that defines the over/under between eternal expansion and eventual recollapse.

Note that ρ_{crit} evolves with H, as do ρ & Ω_m

Only if $\Omega_m = 1$ exactly does it remain 1 for eternity

The Friedmann eqn can be derived from the eqn. of motion of a point on the surface of a uniform expanding sphere, at least in the absence of Λ. It does not depend on scale. So a good first approximation to galaxy formation is to treat the volume that will collapse to form a galaxy as a locally overdense universe with $\Omega_m > 1$.

This is often called a top-hat overdensity



Note that $\vec{x}$ represents all 3 spatial dimensions. The over-dense region is spherical, so Friedmann's eqn applies.

Indeed, as $t \to 0$, $\Omega_m \to 1$, so the curvature and Λ terms may be ignored for the mean $\langle \rho \rangle$.

The solution for the uniform background universe (in which the top-hat is embedded) is

$$\frac{a}{a_0} = \left(\frac{3}{2} H_0 t\right)^{2/3}.$$

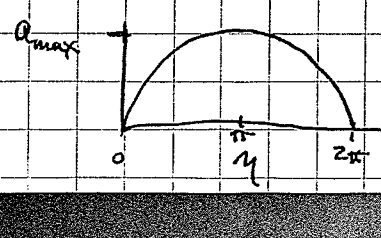
For the top hat, $\Omega_m > 1$, the condition necessary for it to collapse.

But only a little > 1: the initial condition is set by the fluctuations in the CMB when $t \approx 10^5$ yr and $\frac{\delta\rho}{\langle\rho\rangle} \approx 10^{-5}$ — galaxies are thought to arise from the gravitational growth and collapse of these initially tiny over-densities. (Under-densities become voids.)

For $\Omega_m > 1$ the solution is

$$\frac{a}{a_0} = \frac{1}{2} \frac{\Omega_{m,0}}{\Omega_{m,0} - 1} (1 - \cos\eta) \; ; \; H_0 t = \frac{1}{2} \frac{\Omega_{m,0}}{(\Omega_{m,0} - 1)^{3/2}} (\eta - \sin\eta)$$

η is the "development parameter" representing time to recollapse



η runs from 0 to 2π with maximum expansion at $\eta = \pi$

also used in timing argument

Monolithic collapse – hypothesis that the Galaxy formed from the collapse of one big top ~~hat~~.

Historically important, mostly a straw-man first approximation now.

Consider the Milky Way: $M \approx 2 \times 10^{12}\, M_\odot$

In order to gather up that much mass, we need to collect from a volume V

$$M = \rho V \qquad \rho = \Omega_{m,0}\, \rho_{crit} \qquad \rho_{crit} = 1.5 \times 10^{11}\, \frac{M_\odot}{Mpc^3}$$

$$V = \frac{4\pi}{3} R^3_{MW, initial} \qquad \Omega_{m,0} \simeq 0.3$$

corresponding to a current physical radius

$$R_{MW,i} \simeq 2.2 \text{ Mpc}$$

Could also work out volume containing baryonic mass $\Omega_b \simeq 0.04$

Gives $R_{MW} \simeq 1.3$ Mpc

$\left(M_b / M_h < f_b \right)$

This is about a factor a 10 larger than the current radius of the MW encompassing this much mass (~ 200 kpc).

So the collapse factor is ~ 10 in radius

~ 1000 in volume.