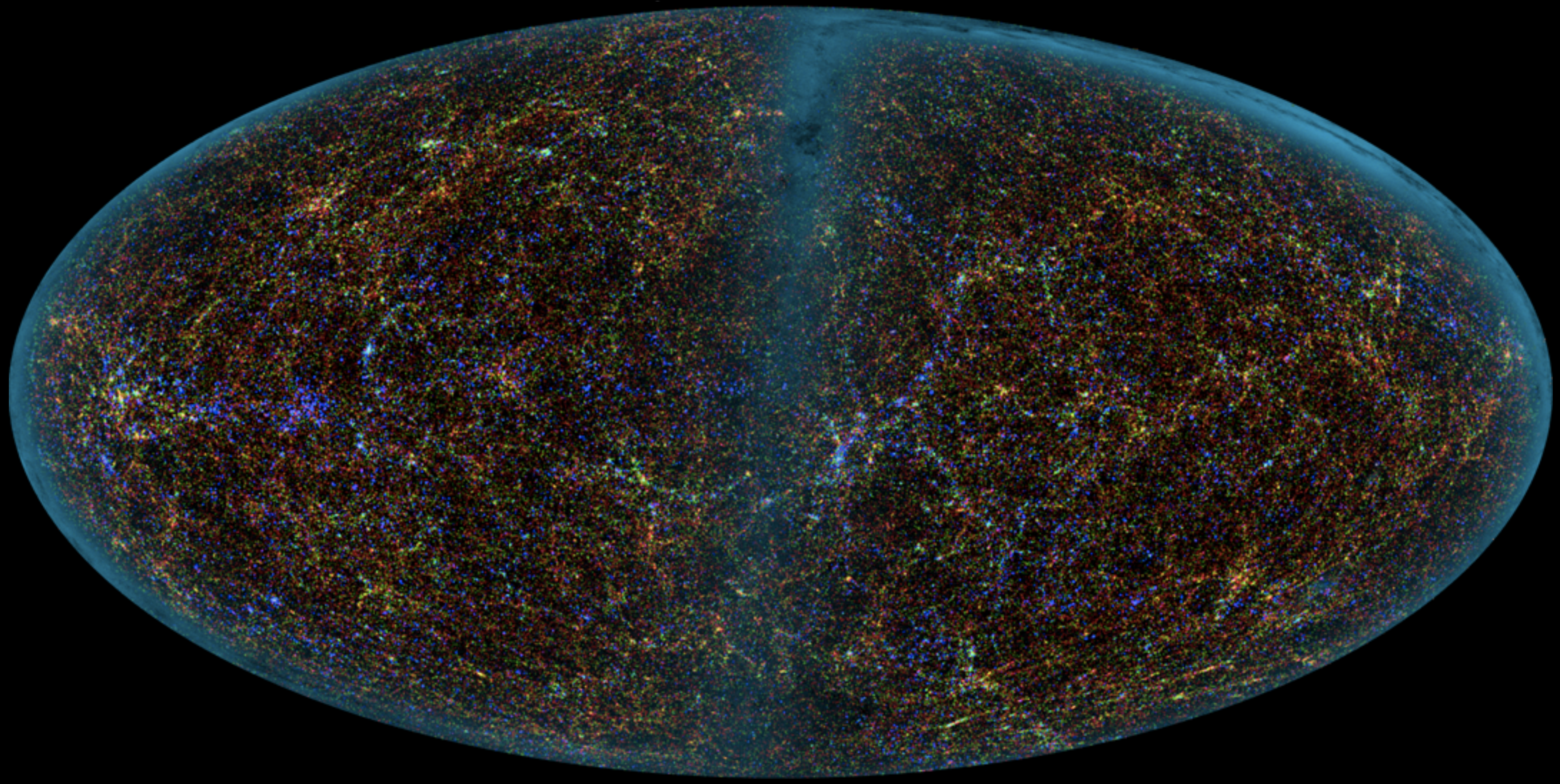
Cosmology

and Large Scale Structure

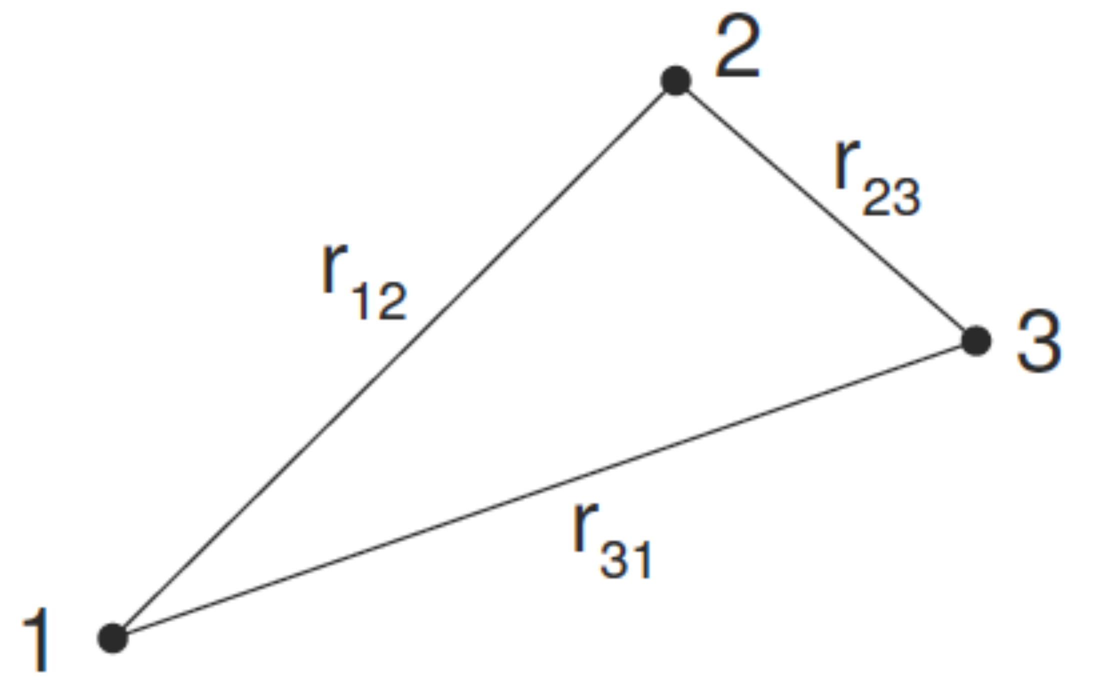


Today
Robertson-Walker Geometry
Friedmann Eqn
Solutions for specific cases

First problem set due

Metrics

geometry in an expanding space-time



3D Euclidean geometry

For Cartesian coordinates (x,y,z) the separation between points is

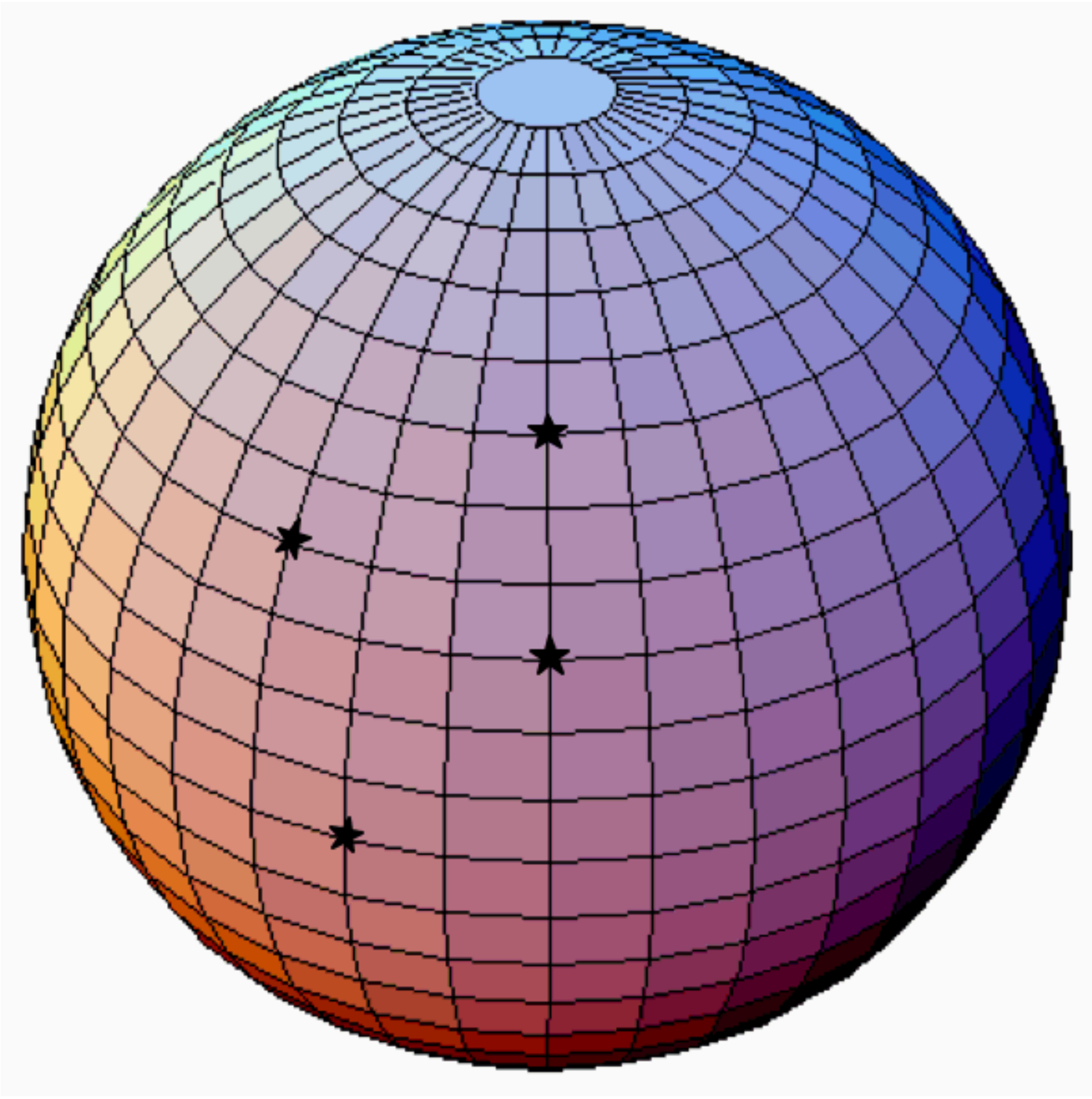
$$d\ell^2 = dx^2 + dy^2 + dz^2$$

For Spherical coordinates (r, θ, ϕ) the separation between points is

$$d\ell^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

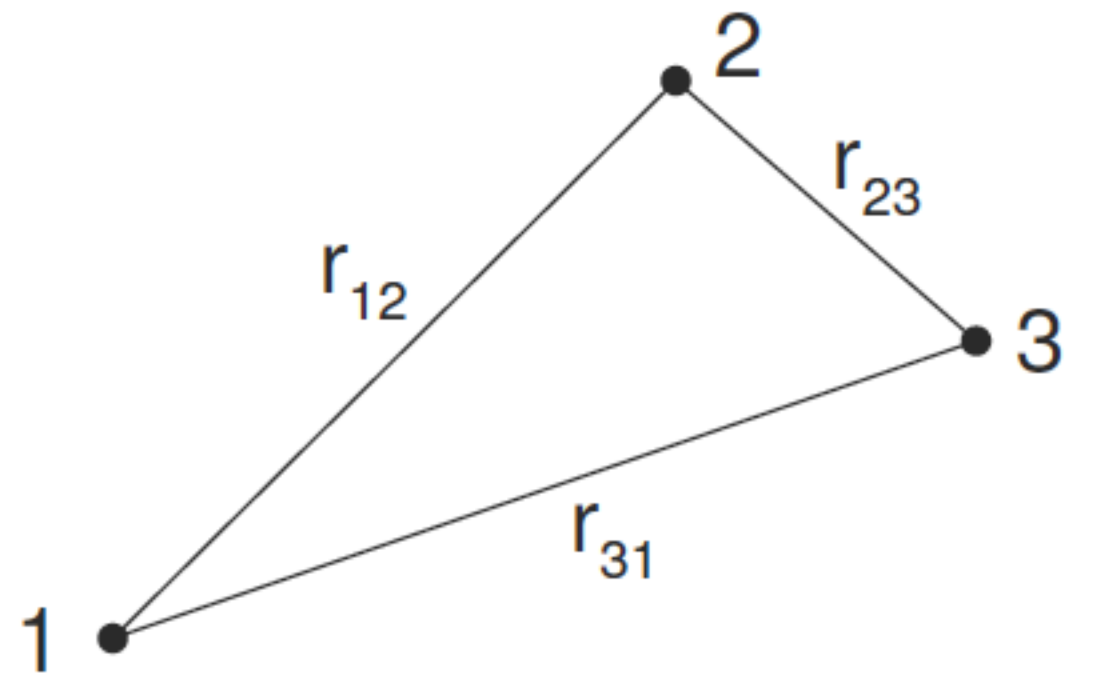
$$d\ell^2 = dr^2 + r^2 d\Omega^2$$

$\sin\theta$ appears because E-W distances are smaller at high latitudes than at the equator.



Metrics

geometry in an expanding space-time



4D Minkowski Spacetime

For Cartesian coordinates (t,x,y,z)
the separation between points is

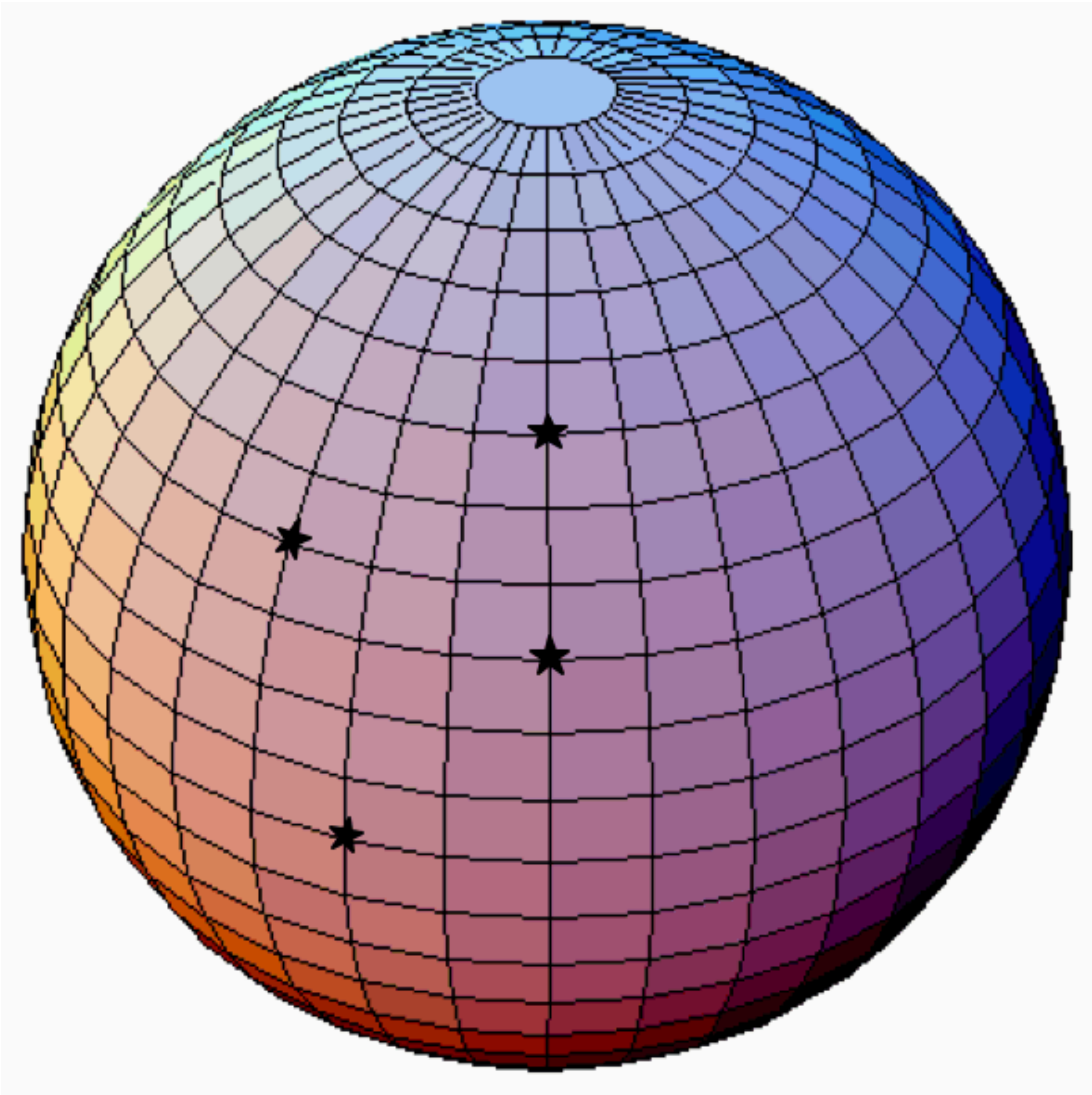
$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$
$$ds^2 = -c^2dt^2 + d\ell^2$$

For Spherical coordinates (t, r, θ, ϕ)
the separation between points is

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\Omega^2$$

For a photon, $ds = 0$

$d\Omega$ is a convenient placeholder for the angular terms that we can usually ignore since we assume homogeneity and isotropy.



Metrics

geometry in an expanding space-time

4D Robertson-Walker Spacetime

Derived from General Relativity
assuming homogeneity and isotropy

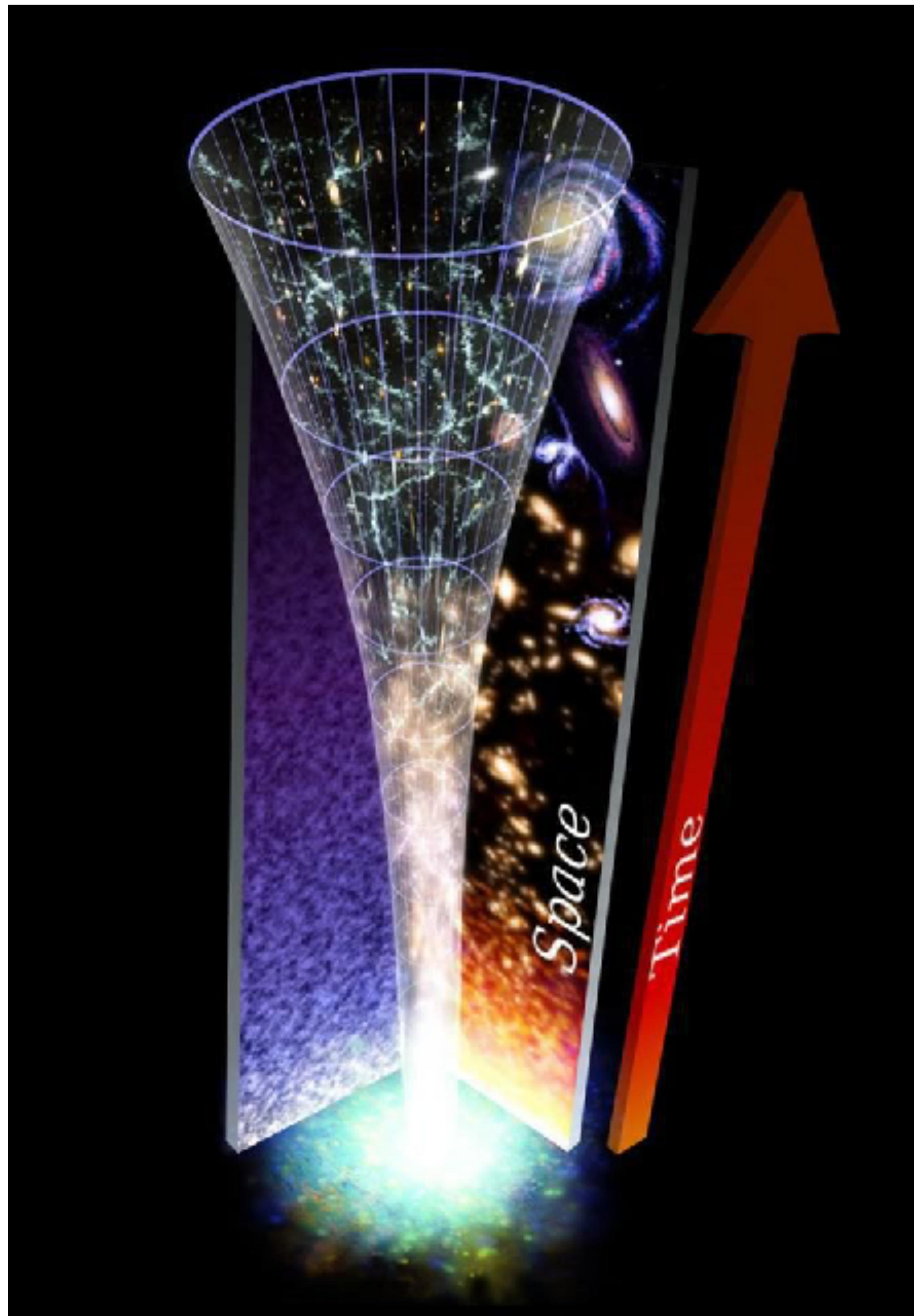
$$ds^2 = -c^2dt^2 + a^2(t)[dr^2 + S_k^2(r)d\Omega^2]$$

Event separation

Expansion factor

Geometric factor

For a photon, $ds = 0$



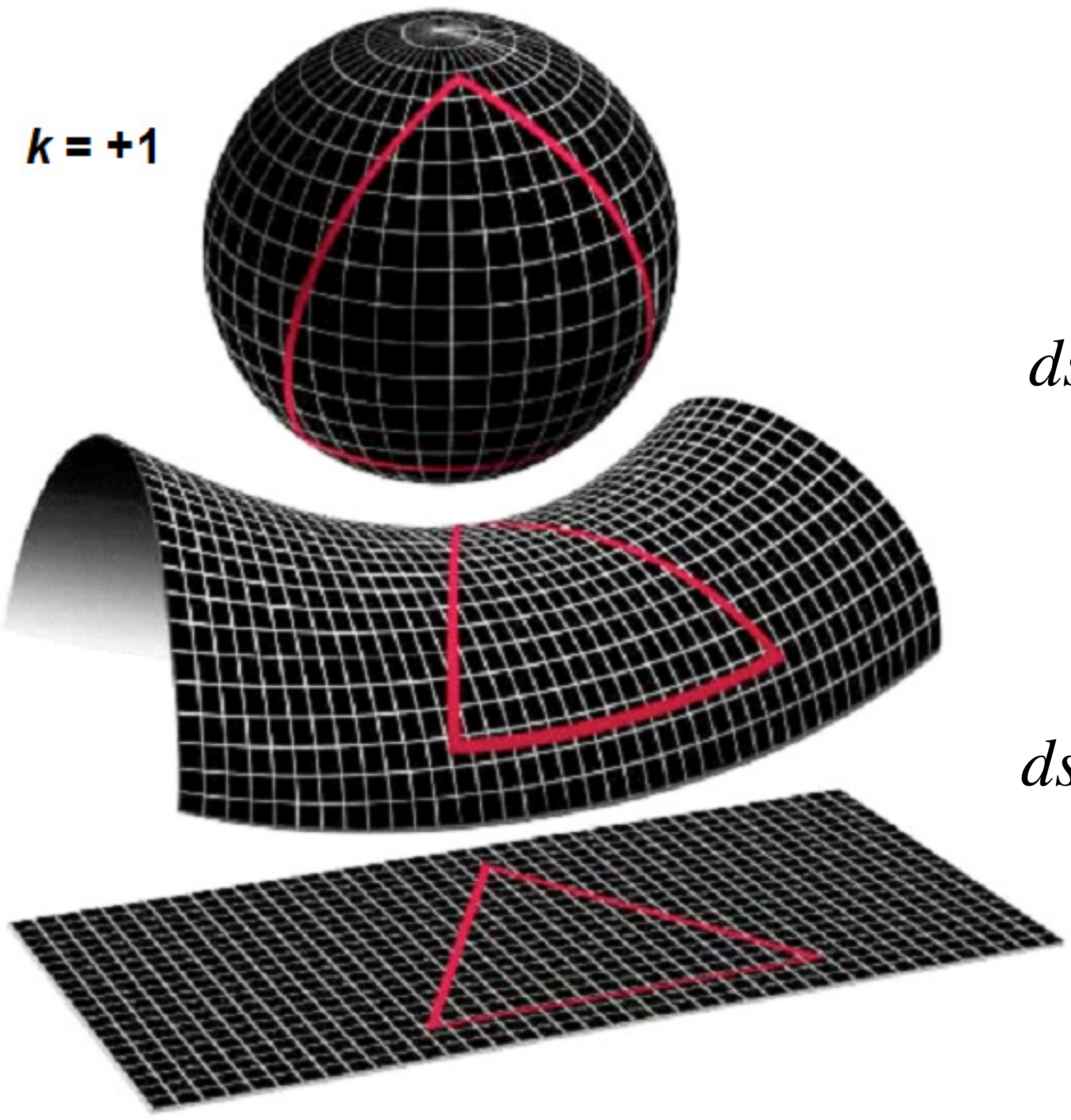
Geometric factor

$$S_k(r) = \begin{cases} R \sin\left(\frac{r}{R}\right) \\ r \\ R \sinh\left(\frac{r}{R}\right) \end{cases} \quad \text{for} \quad \begin{matrix} k = +1 \\ k = 0 \\ k = -1 \end{matrix}$$

$k = +1$ Positively curved

$k = 0$ Flat

$k = -1$ Negatively curved

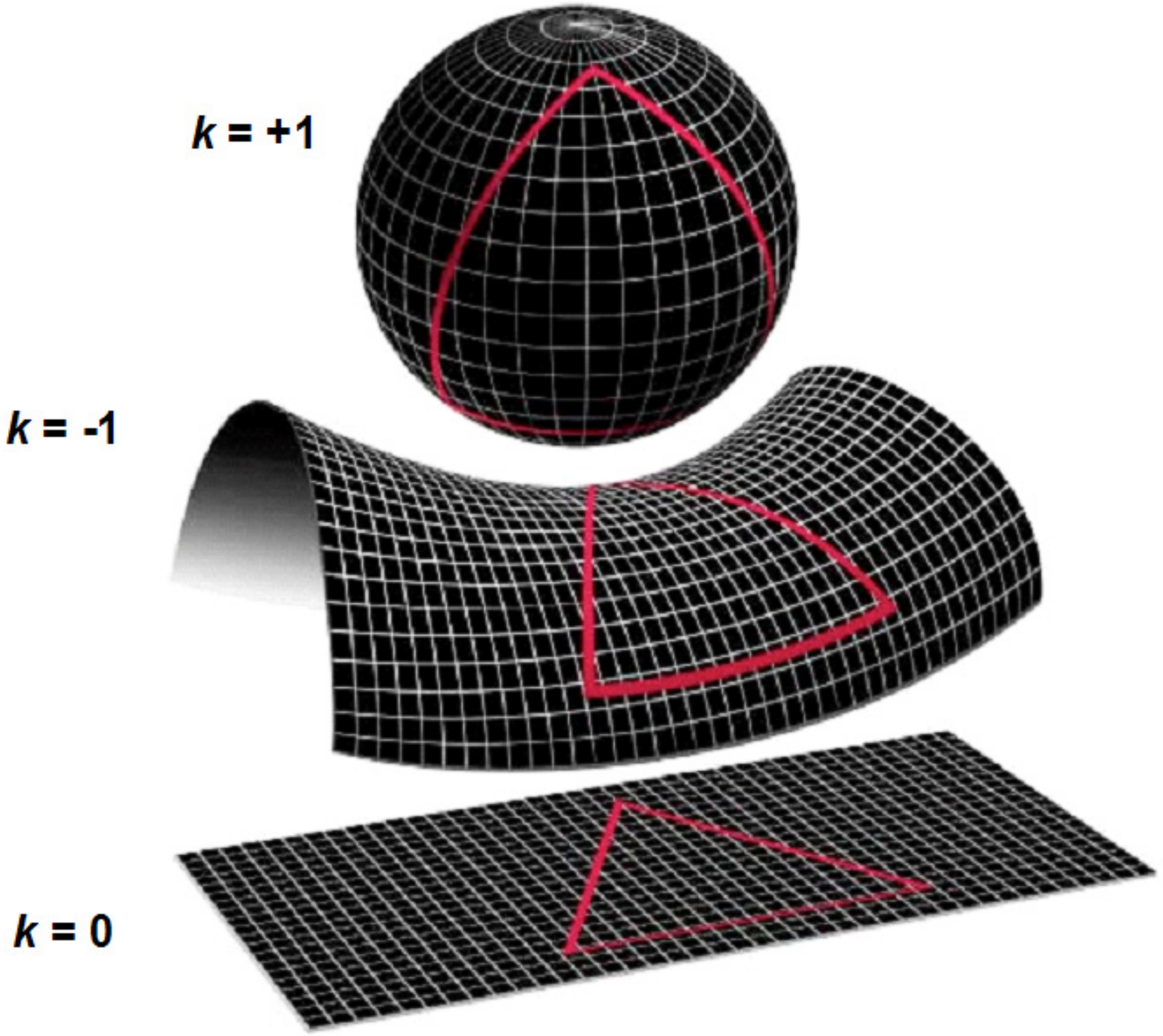


$$ds^2 = -c^2dt^2 + a^2(t)[dr^2 + S_k^2(r)d\Omega^2]$$

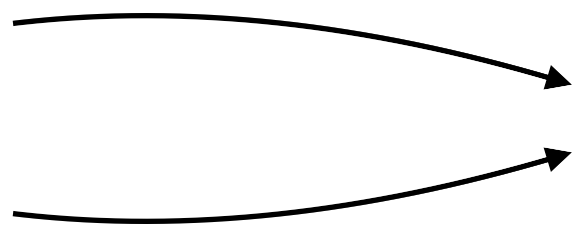
Ryden's notation

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2d\Omega^2\right]$$

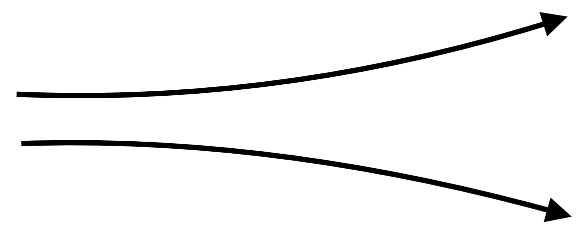
Another common notation swaps (- + + +) convention for (+ - - -) convention; absorbs difference in the definition of the comoving coordinate



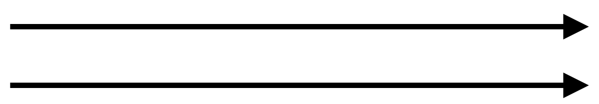
❖ **Positively** curved geometry
Finite total volume
Parallel rays converge
[exceeds critical density; eventually re-collapses]



❖ **Negatively** curved geometry
Infinite volume
Parallel rays diverge
[below critical density; expands forever]



❖ **Flat** geometry
Infinite volume
Parallel rays remain parallel
[exactly critical density; expands forever - just barely]



To get the proper distance to a galaxy we observe, we need to integrate over the expansion since the time of photon emission:

$$c\,dt = a(t)\,dr$$

$$c\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^r dr = r$$

where we make use of the fact that for photons, $ds = 0$

we know the expansion factor from the redshift

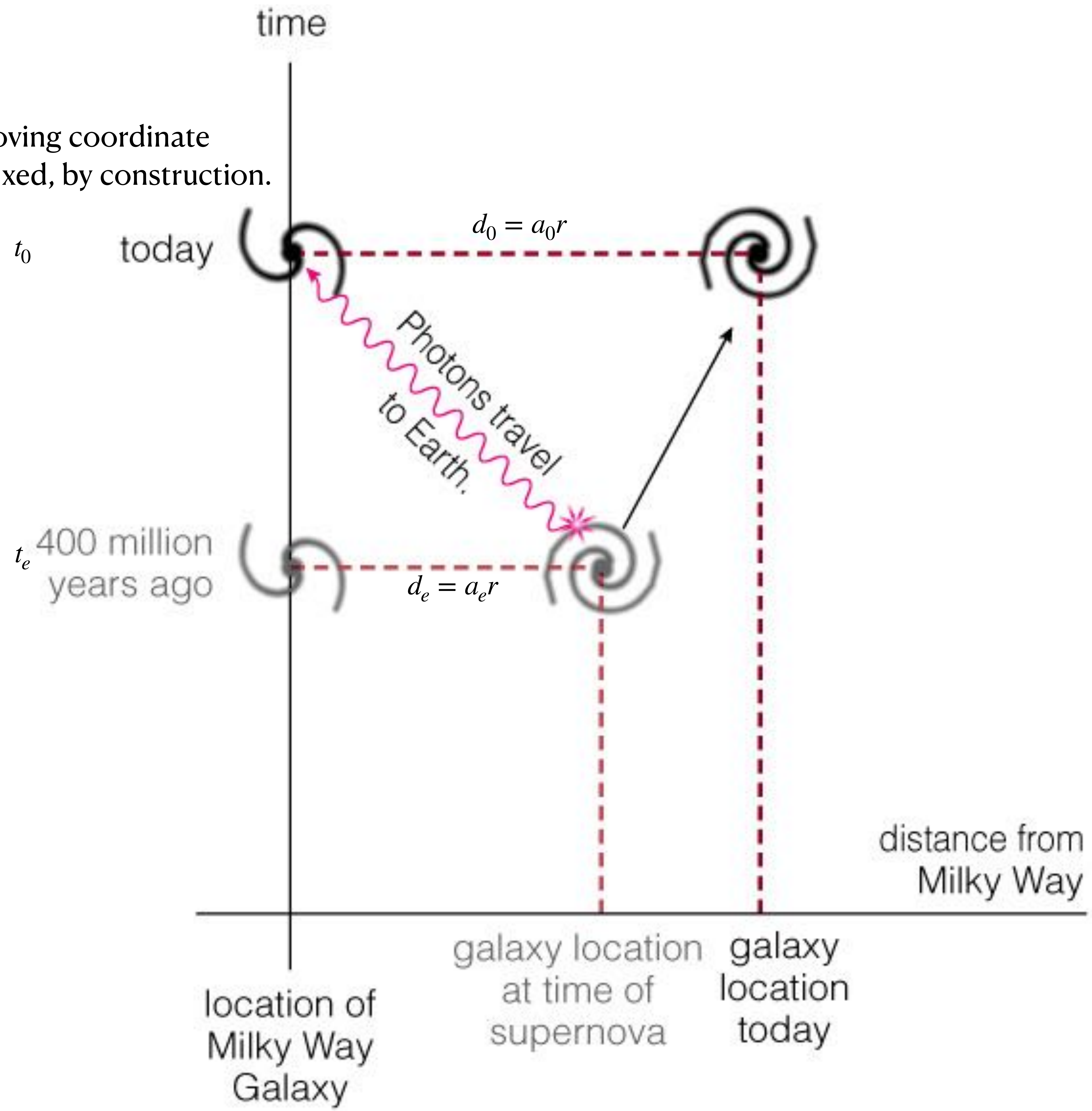
$$\frac{a(t_0)}{a(t_e)} = \frac{1}{a(t_e)} = 1 + z$$

t_0 is now, so by construction $a(t_0) = 1$.

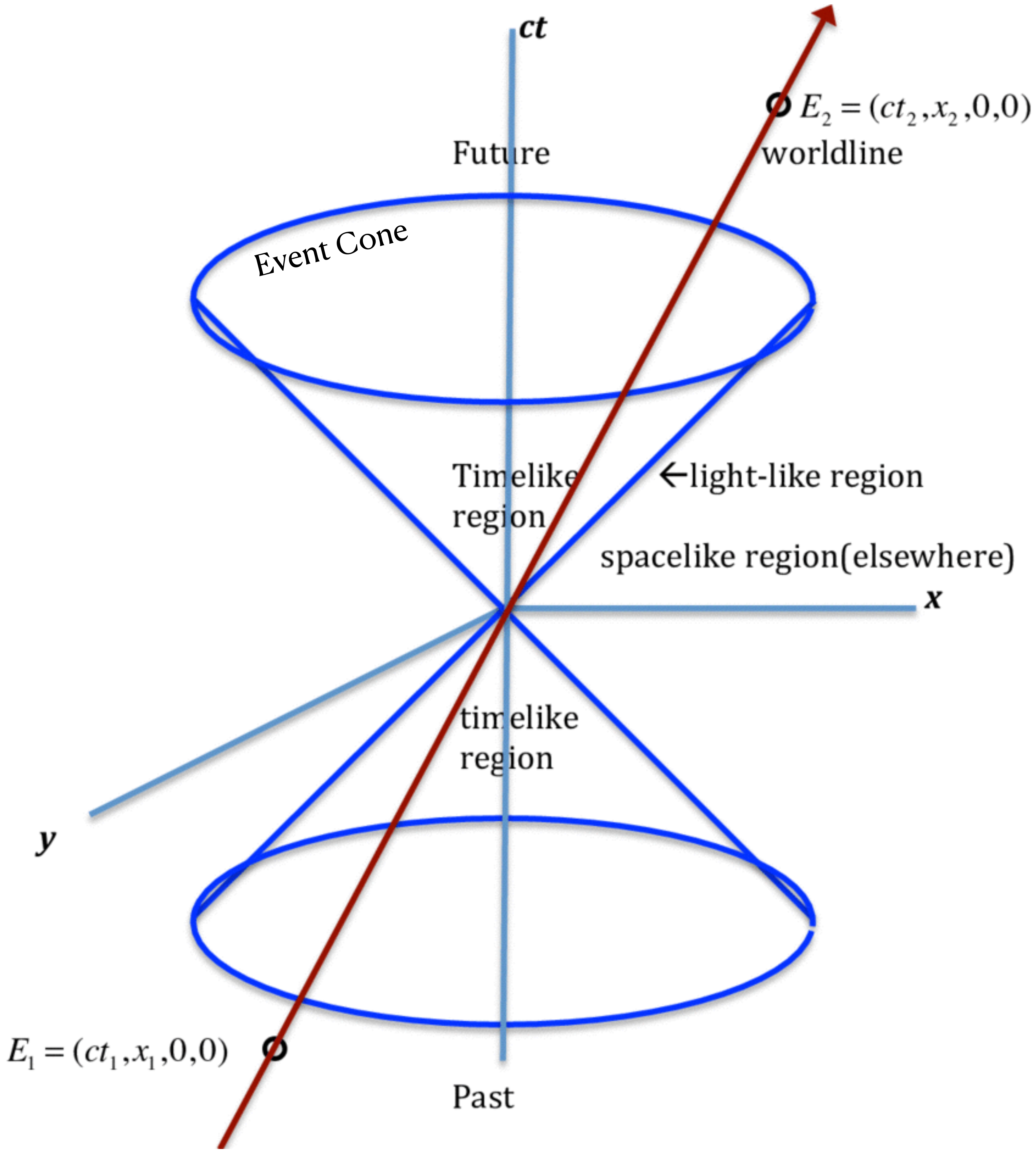
Cosmological parameters specify $a(t)$ through solution of the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{(aR_0)^2} + \frac{c^2}{3}\Lambda$$

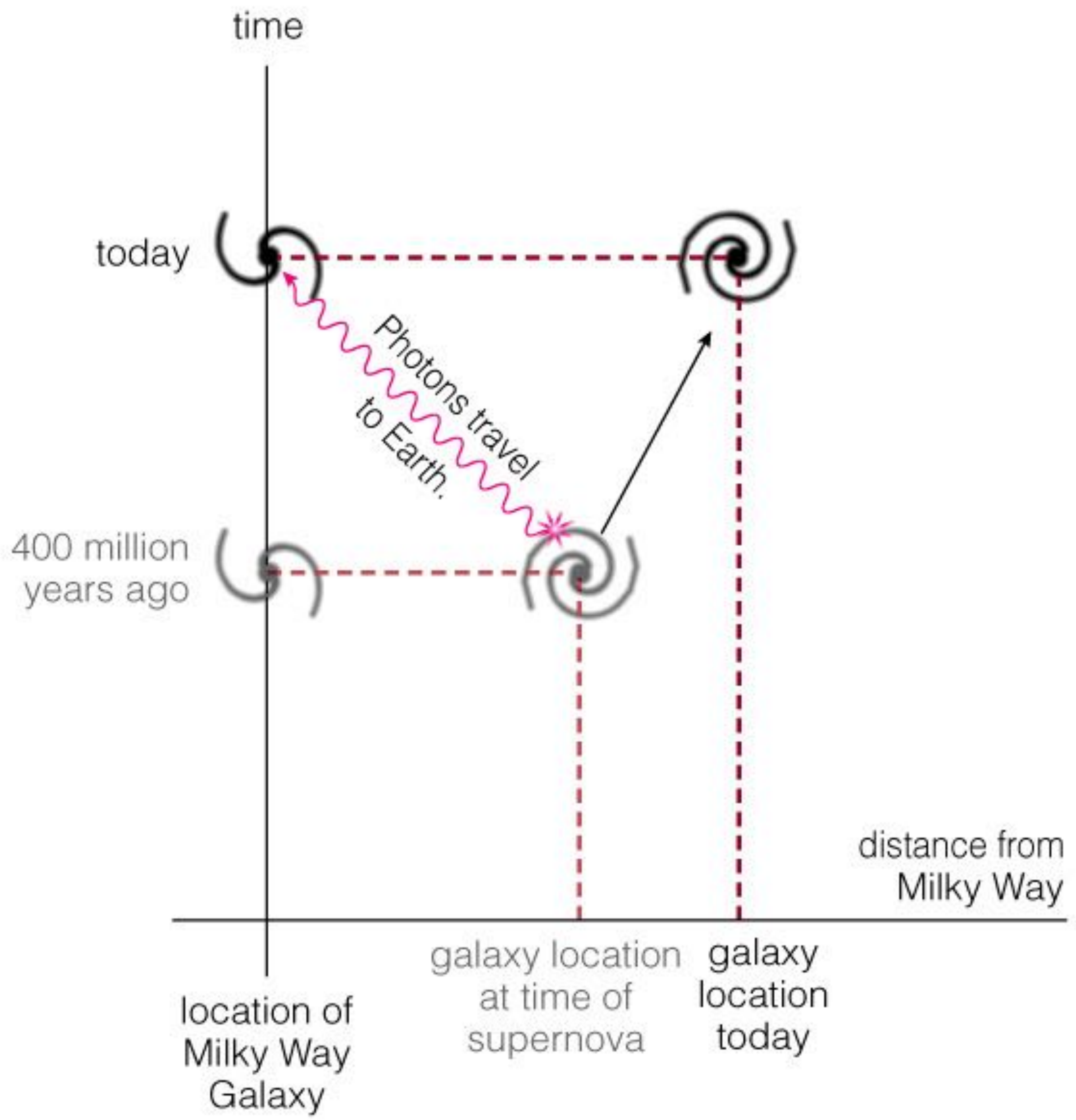
The comoving coordinate separation is fixed, by construction.



Spacetime diagram



Spacetime diagram
for two galaxies



$c^2dt^2 > d\ell^2$ Time like region - cone of causal connectivity

$c^2dt^2 = d\ell^2$ Light like cone - traversed by photons in vacuum

$c^2dt^2 < d\ell^2$ Space like region - out of causal contact

Model Universes

governed by

Einstein field equation

$$\mathscr{R}_{\mu\nu} - \frac{1}{2}\mathscr{R} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda g_{\mu\nu}$$

which bequeath us the

Roberston-Walker metric

$$ds^2 = -c^2 dt^2 + a^2(t)[dr^2 + S_k^2(r) d\Omega^2]$$

mostly just care about
$c\,dt = a(t)\,dr$
for events tied to the comoving coordinate system

and the

Friedmann equation

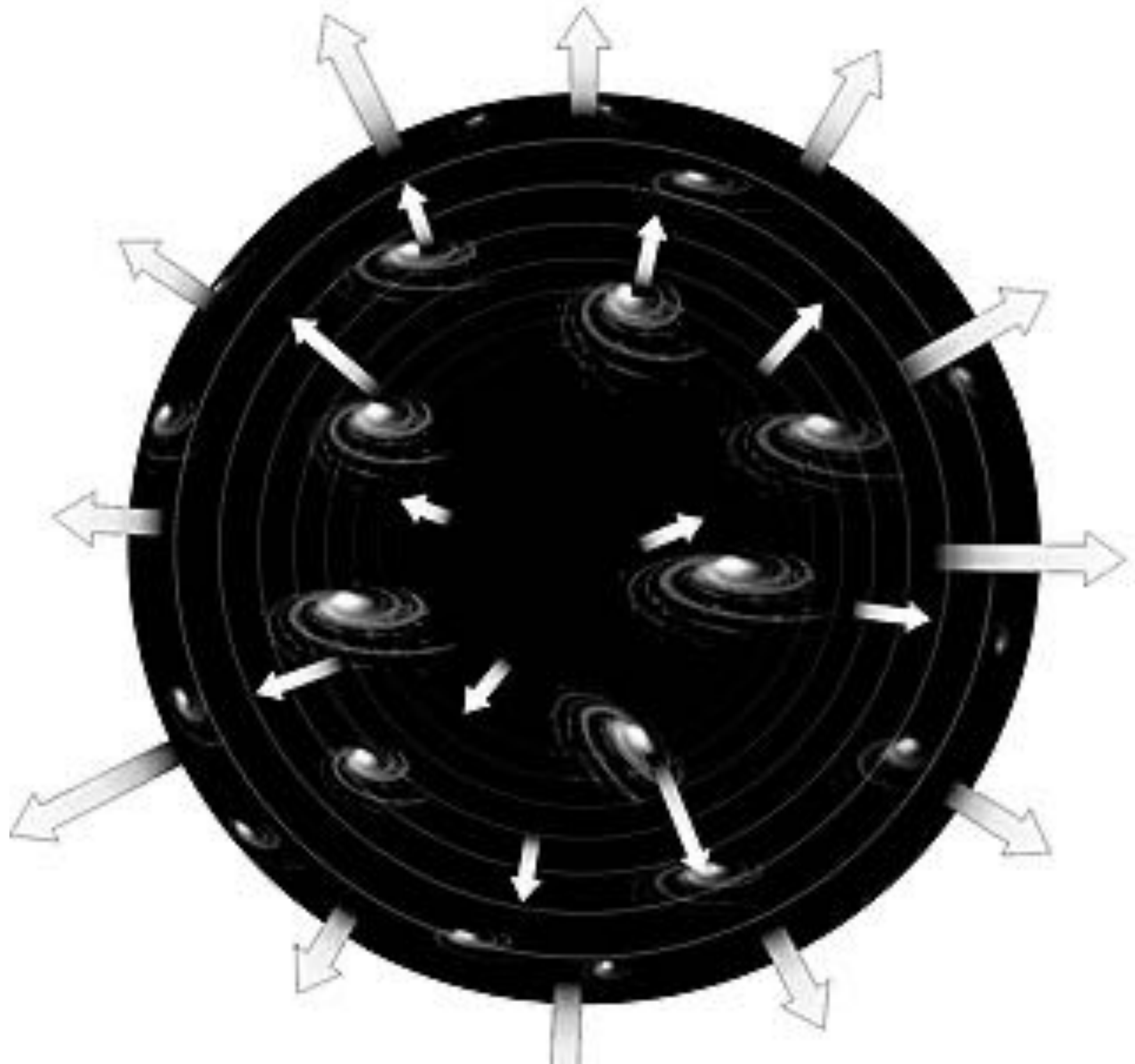
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{(aR_0)^2} + \frac{c^2}{3}\Lambda$$

expansion rate

gravitating mass-energy

geometry
R_0 is the radius of curvature

cosmological constant



Friedmann equation from GR

Ignoring the cosmological constant for the moment,

$$\mathscr{R}_{\mu\nu} - \frac{1}{2}\mathscr{R} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$T_{\mu\nu} = \begin{bmatrix} \rho & & & 0 \\ & P & & \\ & & P & \\ 0 & & & P \end{bmatrix}$$

Ricci Tensor | Stress-Energy Tensor

no rotationally invariant 4-vectors $\mathscr{R}_{ti} = 0$ $T_{ti} = 0$

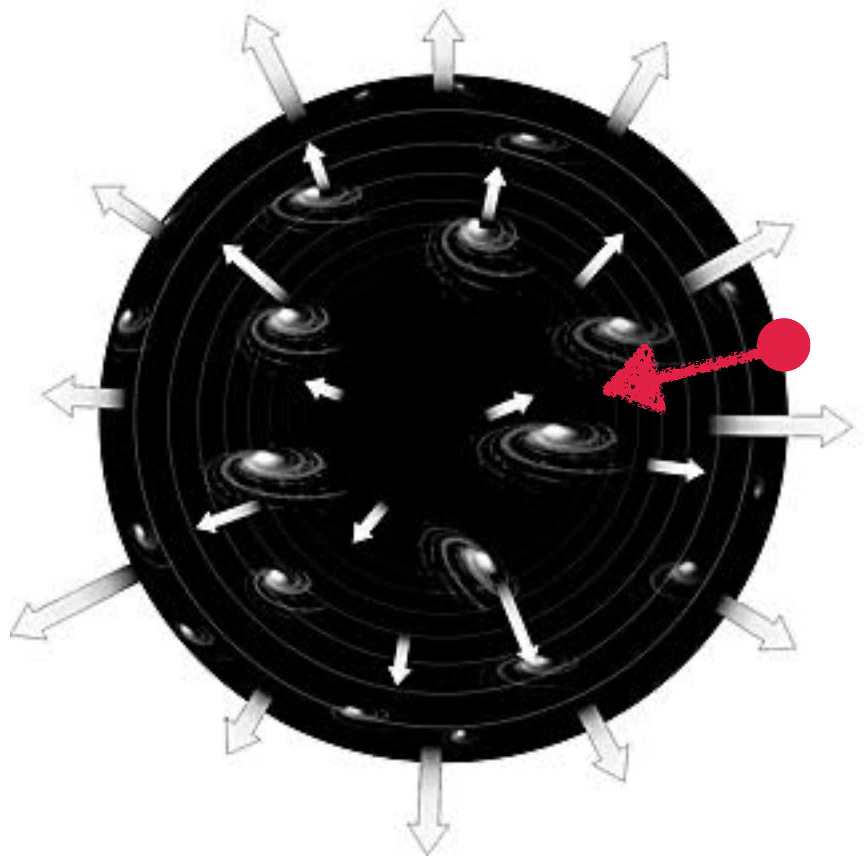
curvature of space related to the gravitational potential $\mathscr{R}_{tt} = U(t)$ $T_{tt} = \rho(t)$ mass density

Poisson equation in Newtonian gravity

$$\nabla^2 \Phi = 4\pi G \rho$$

energy gravitates too $\mathscr{R}_{ij} = g_{ij} V(t)$ $T_{ij} = g_{ij} P(t)$ pressure stemming from the energy density in relativistic components

Friedmann equation from GR



$$U(t) = -3\frac{\ddot{a}}{a} \qquad = 8\pi G(\frac{1}{2}\rho + \frac{3}{2}P)$$

The units of density and pressure are taken to be the same here; they are related by c² in conventional units — pressure is an energy density.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

Looks like the Newtonian acceleration of a point on the surface of a sphere with a term added for pressure.

Aside: Newtonian Cosmology

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho$$

$$\ddot{a} = -\frac{GM}{a^2}$$

$$\ddot{a} = -\frac{4\pi G}{3}\rho a \qquad \text{for} \qquad M = \frac{4\pi G}{3}\rho$$

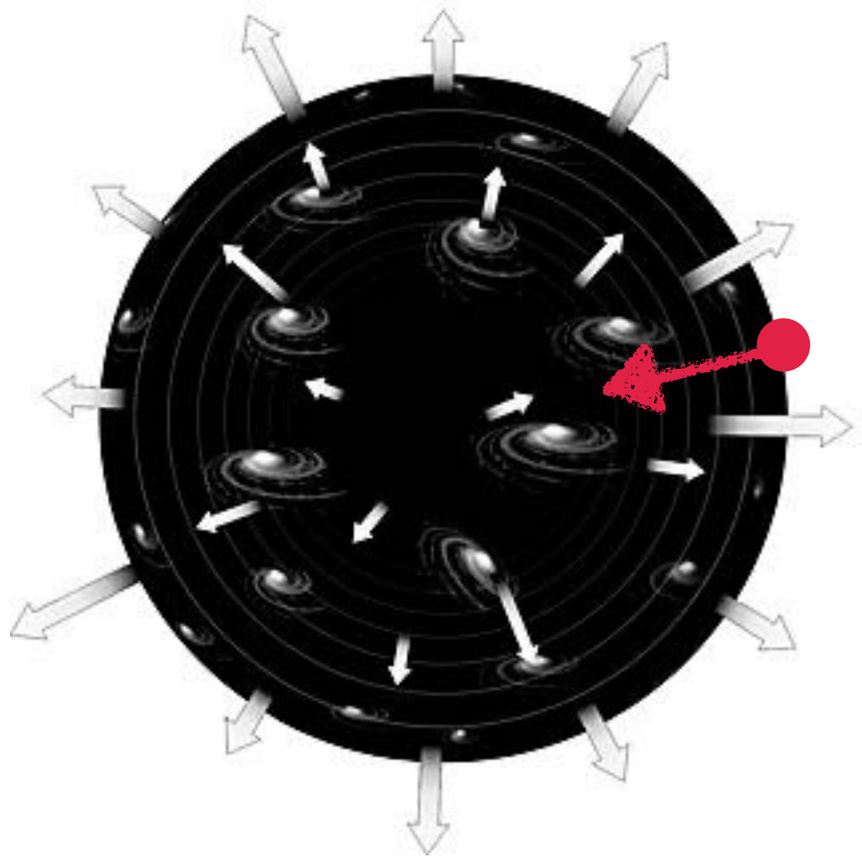
ρ uniform, by assumption (homogeneity)

Pick up integration constant with units of energy, *U*. This determines whether the expansion is “bound” or not - i.e., the sphere recollapses or expands forever.

$$\dot{a}^2 - \frac{8\pi G}{3}\rho a^2 = 2U$$

The behavior is the same for any uniform sphere, so its size ℓ is irrelevant

Friedmann equation from GR



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Looks like the Newtonian acceleration of a point on the surface of a sphere with a term added for pressure.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho$$

$$\mathcal{R}_{ij} = g_{ij}V(t)$$

$$V(t) = a\ddot{a} + 2\dot{a} + 2k \quad = 4\pi G(\rho - P)a^2$$

Combine equations for *U(t)* and *V(t)* to eliminate the second time derivative, leaving the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{(aR_0)^2}$$

The book replaces mass density with energy density (eq. 4.20)

$$\rho = \frac{\varepsilon}{c^2}$$

Friedmann equation

Including the cosmological constant, the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{(aR_0)^2} + \frac{c^2}{3}\Lambda$$

Energy conservation $D_\mu T_{\mu\nu} = 0$ in an expanding universe

$$\frac{d}{da}(a^3\rho) = -\,a^2 P$$

just PdV work

total mass-energy "work" done in expansion

Solutions depend on the equation of state $P = w\rho$

$w = 0$ non-relativistic mass ("dust")

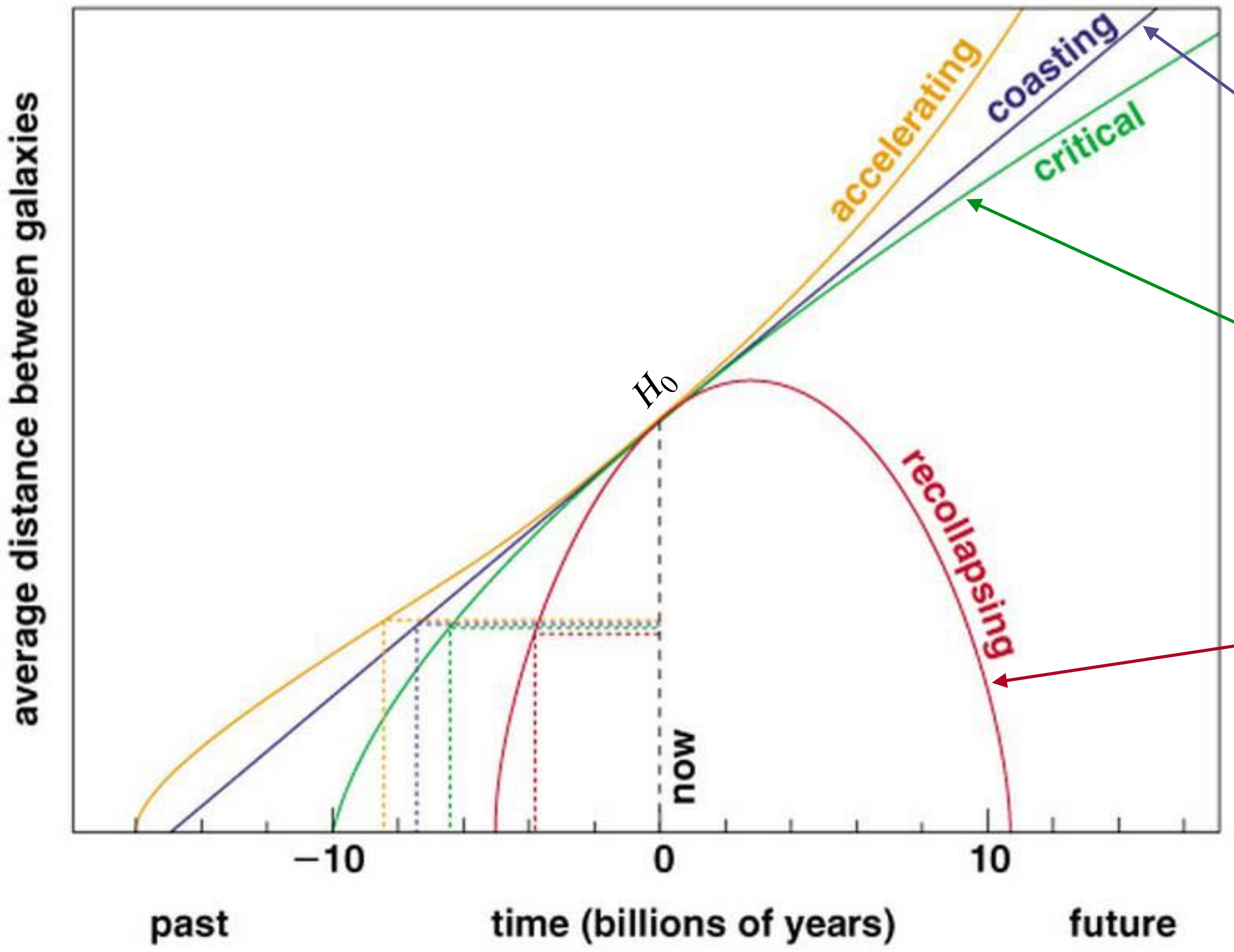
$w = \frac{1}{3}$ photons

$w = -\,1$ cosmological constant

The cosmological constant is repulsive because of the negative sign in its equation of state.

Possible expansion histories

$a(t)$



t

Dust models (i.e, normal matter)

$w = 0$

Dust models of different densities

Coasting aka Empty Universe $\rho = 0$

Critical aka Einstein-de Sitter Universe $\rho = \rho_c = \dfrac{3H_0^2}{8\pi G}$

Recollapsing aka Phoenix Universe $\rho > \rho_c$

de Sitter Universe

$\rho = 0$

$w = -1$

purely accelerated expansion

The expansion started by the Big Bang
is resisted by the attraction of gravity.

The more dense the universe, the more gravity...
a balance is reached at a critical density:

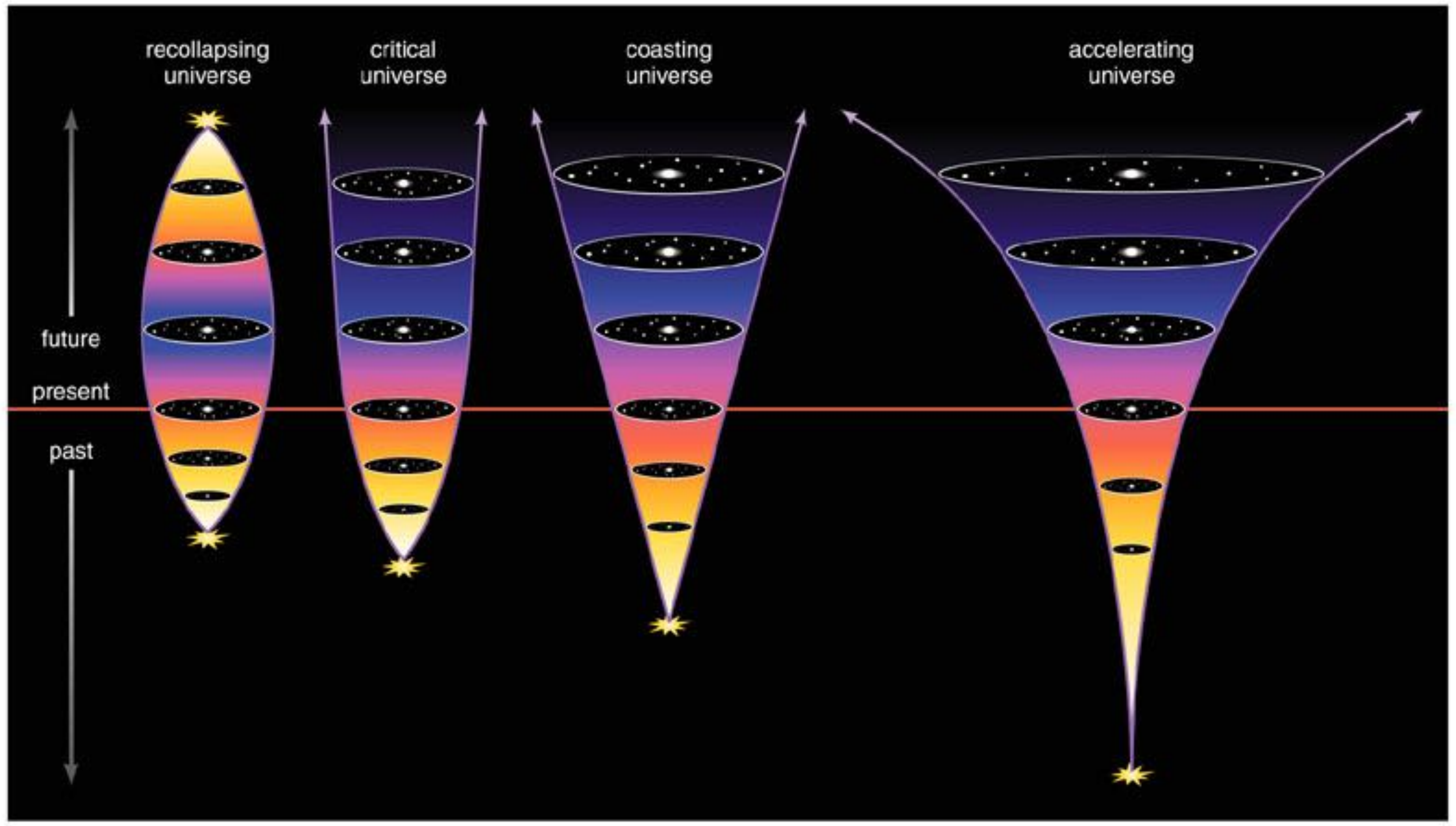
	IF	the universe is
$k = -1$	$\rho < \rho_{crit}$	OPEN: expands forever
$k = 0$	$\rho = \rho_{crit}$	FLAT
$k = +1$	$\rho > \rho_{crit}$	CLOSED: eventually re-collapses

These are the traditional options in the absence of a repulsive force like a cosmological constant/dark energy

Density is destiny

It determines the age, geometry, and ultimate fate of the universe

time



CLOSED

CRITICAL

OPEN

ACCELERATING

$k = 0$

FLAT

Density = Critical

$k = +1$

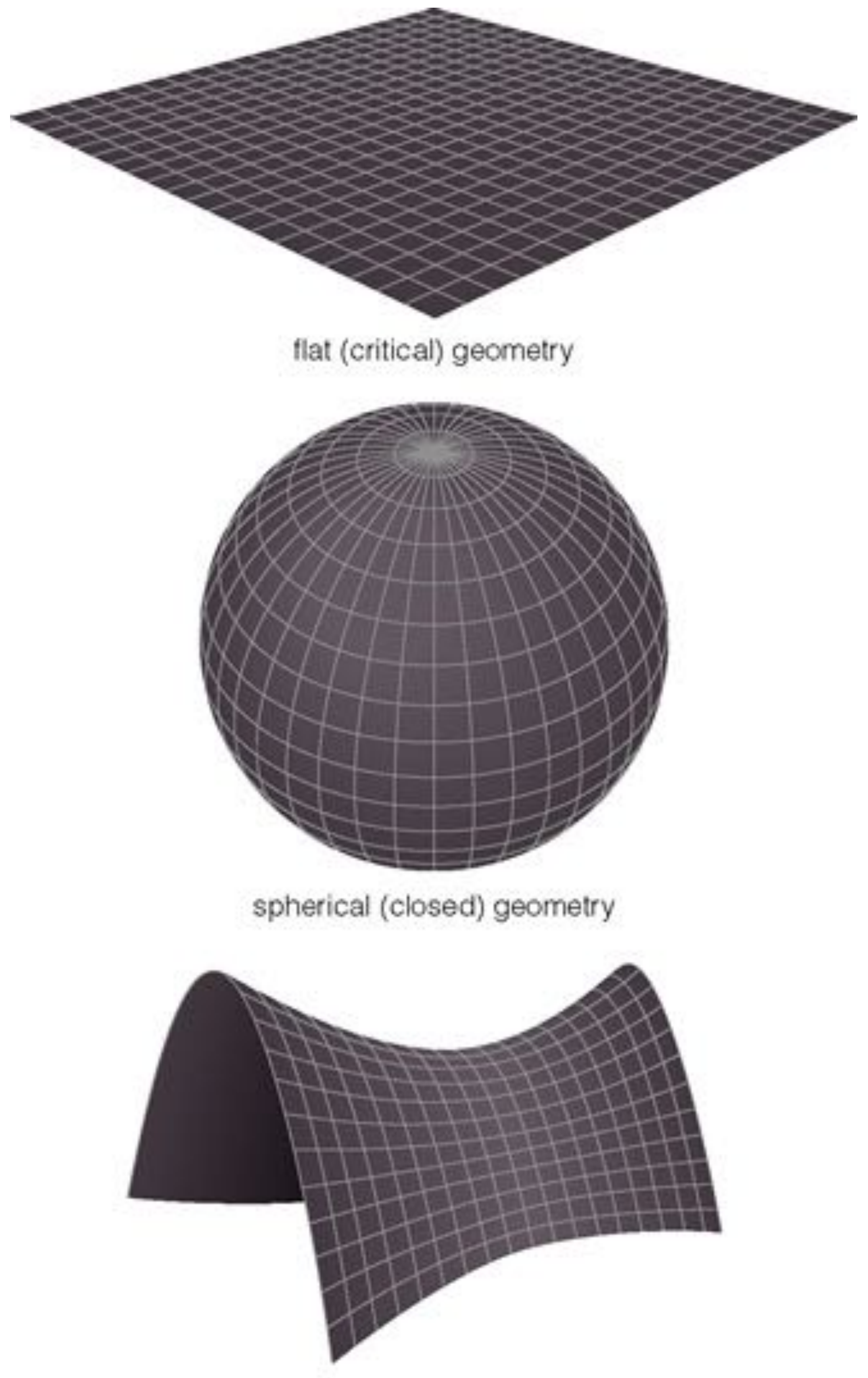
CLOSED

Density > Critical

$k = -1$

OPEN

Density < Critical



Space can be curved.

The overall geometry of the universe is determined by the total density of matter and energy.

Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{(aR_0)^2} + \frac{c^2}{3}\Lambda$$

can be written

$$H^2 = H^2(\Omega_m + \Omega_r + \Omega_k + \Omega_\Lambda)$$

where

$H \equiv \frac{\dot{a}}{a}$ does not remain constant, so the Hubble "constant" is just the current value of the Hubble parameter H(z).

$$\Omega_m = \frac{8\pi G}{3H^2}\rho$$

mass density

$$\Omega_r = \frac{\varepsilon\, c^{-2}}{\rho_c}$$

radiation density

$$\Omega_k = -\frac{kc^2}{(aR_0H)^2}$$

curvature

Flat cosmologies have $k = 0$ so $\Omega_k = 0$

$$\Omega_\Lambda = \frac{c^2\Lambda}{3H^2}$$

cosmological constant

Λ is constant but Ω_Λ evolves as *H* evolves

the sum of density parameters so defined must be unity: $\Omega_m + \Omega_r + \Omega_k + \Omega_\Lambda = 1$