complete discussion of Bias

Virial Theorem

Following Bothun p. 4.1.1

Moment of Inertia of system of N particles

$$I = \sum_{i=1}^{N} m_i r_i^2 = \sum_{i=1}^{N} m_i (x_i^2 + y_i^2 + z_i^2)$$

time derivative of moment of inertia

$$\frac{dI}{dt} = \dot{I} = \sum_{i=1}^{N} m_i (2x_i\dot{x}_i + 2y_i\dot{y}_i + 2z_i\dot{z}_i)$$

$$\frac{d^2I}{dt^2} = \frac{1}{2}\ddot{I} = \underbrace{\sum_{i=1}^{N} m_i(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}_{mv^2 = \text{kinetic energy, } T = \frac{1}{2}mv^2} + \underbrace{\sum_{i=1}^{N} m_i(x_i\ddot{x}_i + y_i\ddot{y}_i + z_i\ddot{z}_i)}_{\vec{r}\cdot(m\vec{a}) \text{ potential energy, } W}$$

in general

$$\boxed{\frac{1}{2}\ddot{I} = 2T + W}$$

After a few dynamical times, a self-gravitating system will obtain VIRIAL EQUILIBRIUM (become "virialized") in which the time-averaged moment of inertia is steady:
$\langle \dot{I} \rangle \rightarrow$ constant so $\langle \ddot{I} \rangle \rightarrow 0$ [Note - this is violated during mergers, reinstated afterwards.] — NON-STATIONARY SYSTEM when $\dot{W} \neq 0$

$$\boxed{\frac{1}{2}\langle \ddot{I} \rangle = 0 = 2\langle T \rangle + \langle W \rangle}$$ VIRIAL THM

For self gravitating system of particles,

$$W = -\frac{G}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{m_i m_j}{r_{ij}}$$

— correct for double counting

IFF all m_i the same (or nearly so), then can replace sums →

$$\sum_i^N \sum_j^N m_i m_j = N(N-1) m_i^2 \rightarrow M^2 \quad \text{for large } N$$

(where M is the total mass)

so $$W = -\frac{GM^2}{2R_{hms}}$$ where R_{hms} is the harmonic mean separation

$$\frac{1}{R_{hms}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{r_i}$$

OK, so now use this: $2\langle T\rangle + \langle W\rangle = 0 \rightarrow \langle T\rangle = -\frac{1}{2}\langle W\rangle$

$T > 0$
$W < 0$

$T = \frac{1}{2} N v_i^2 m_i$

$\langle T\rangle = \frac{1}{2} M \langle v_i^2\rangle$ $\qquad \sigma_v = \langle v_i^2\rangle^{1/2}$ $\qquad \sigma_v$ rms velocity dispersion

$$2\langle T\rangle + \langle W\rangle = 0 = \cancel{2}\,\cancel{\tfrac{1}{2}} M\sigma_v^2 - \frac{1}{2}\frac{GM^2}{R_{hms}}$$

W negative!

$$\cancel{M}\sigma_v^2 = \frac{1}{2}\frac{GM^{\cancel{2}}}{R_{hms}}$$

$$\boxed{M = \frac{2\sigma_v^2 R_{hms}}{G}}$$ VIRIAL MASS ESTIMATOR

σ_v 3D velocity dispersion – take care in relating this to L.O.S measured σ_{LOS} – $\sigma_{LOS} = \sigma_v$ ONLY IF system isotropic

need to measure σ_v, R_{hms} to get mass

In practice, $R_{hms} \approx 1.25\, R_{eff}$ is a good approximation

R_{eff} = half light radius → hence common assumption "light traces mass"

i.e., $R_{eff}(\text{light}) = R_{eff}(\text{mass})$

This is a more serious issue than $R_{hms} \approx 1.25 R_{eff}$ approximation

σ_x σ_y σ_z

$\sigma_{LOS}^2 = \sum_j (\sigma_j / \sin^2 i_j)^2$

Applied to clusters to get M

$\rightarrow M/L \rightarrow M/L \times j \rightarrow \Omega_m$

appa
Sou
posit

true
source
position

$$\frac{t_{Relax}}{t_{cross}} \approx \frac{N}{6\ln(N/2)}$$

so whole galaxy w/ $N = 10^{11}$ $t_R \gg$ universe
for a Globular w/ $N \sim 10^6$ $t_R \sim 10^{10}$ yr

Useful time scales

SG chapter 3

Question: what is crossing time of a Globular C? The MW? The "virial" DM halo?

crossing time: $t_c = \frac{2R}{\sigma}$
(how long it typically takes to go from one side of a system to the other)

dynamical time: $t_d = \sqrt{\frac{3\pi}{16 G\rho}}$
typical orbital time [derived for homogeneous sphere of density ρ]

relaxation time: $\frac{t_R}{t_c} = \frac{N}{48 f^2}$ N objects carrying a fraction f of the total mass

$\therefore$ Relaxation time long when N large

$\boxed{t_R/t_c \approx N/6\ln(N/2)}$

relaxation defines how long it takes to "forget" initial condition,
i.e. randomize initial vector

Strong encounters NOT relevant (S.G. 3.2)

imagine a star sweeping out a cylinder of radius b (impact parameter)

An interaction is "strong" if $b < r_s = \frac{2Gm}{v^2}$

So that ΔP.E. $\sim$ K.E. (Assuming all stars have same mass)

$$\frac{GM^2}{r} \gtrsim \frac{1}{2} m v^2$$

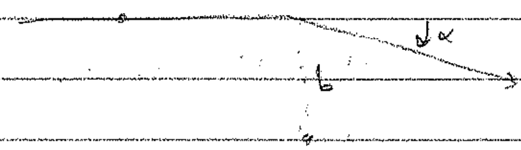
For $v \simeq 30\ \mathrm{km\,s^{-1}}$ ($\simeq \sigma$) & typical $m = 0.5 M_\odot$, $r_s \approx 1$ AU [SMALL compared to separation]

time it takes to encounter another star
depends on volume of cylinder: $\pi r_s^2 (V t_s)$

$$t_s = \frac{v^3}{4\pi G^2 m^2 n} \approx 4\times10^{12}\,\mathrm{yr} \left(\frac{V}{10\ \mathrm{km\,s^{-1}}}\right)^3 \left(\frac{m}{M_\odot}\right)^{-2} \left(\frac{n}{1\ \mathrm{pc^{-3}}}\right)^{-1}$$

$t_s \sim 10^{15}$ yr in solar neighborhood : $\gg$ age of universe ($\sim 10^{10}$ yr)

For weak encounters, we can use the impulse approximation, wherein there is no change to $|\vec{V}|$ and only a tweak to the direction:



$$\alpha = \frac{\Delta V_\perp}{V} = \frac{2Gm}{bV^2}$$

Cumulatively

$$\langle \Delta V_\perp^2 \rangle = \int_{b_{min}}^{b_{max}} nVt \left(\frac{2Gm}{bV}\right)^2 2\pi b \, db = \frac{8\pi G^2 m^2 n t}{V} \ln\left(\frac{b_{max}}{b_{min}}\right)$$

THE BIG FUDGE: $\Lambda = \dfrac{b_{max}}{b_{min}}$

$b_{min} = r_s$

b_{max} = "size of system"

$$t_{relax} = \frac{V^3}{8\pi G^2 m^2 n \ln\Lambda} = \frac{t_s}{2\ln\Lambda}$$

$\Lambda \gg 1$ so $t_{relax} < t_s$

WEAK ENCOUNTERS RELAX SYSTEM

Typically $\ln\Lambda \sim 20$ but this is a huge fudge