The fundamental way we characterize the morphology of galaxies is through their surface brightness profiles. The light from galaxies follows a distribution of brightness on the night sky given by $I(x, y)$ where $I$ is the intensity or surface brightness distribution measured in units of luminosity per unit area at position $(x, y)$ where the area is either in physical units $\left(p c^{2}\right)$ or an angular area in square arcseconds. The intensity is often represented by a radially-averaged function, $I(R)$. The total luminosity of a galaxy is then just:

$$
\begin{equation*}
L_{t o t}=2 \pi \int_{0}^{\infty} I(R) R d R \tag{1}
\end{equation*}
$$

Astronomers usually quote surface brightness in units of magnitudes per square arcsec denoted by the symbol $\mu$. The quantity $\mu$ represents the apparent magnitude of the equivalent total light observed in a square arcsecond at different points in the distribution. It can be related to the physical surface brightness profile through:

$$
\begin{equation*}
\mu=-2.5 \log _{10} I+C \tag{2}
\end{equation*}
$$

If $I$ is measured in $L_{\odot} \mathrm{pc}^{-2}$ then the constant $C$ can be found by going back to the distance modulus formula and determining the amount of light in a square arcsec for a galaxy observed at distance $d$ (in pc) i.e. the light in a sq. arcsec is:

$$
\begin{equation*}
L=I d^{2} \delta \theta^{2} \tag{3}
\end{equation*}
$$

where $\delta \theta=1 "=1 / 206265$ radians. The distance modulus formula is:

$$
\begin{equation*}
m=M+5 \log _{10} d /(1 p c)-5 \tag{4}
\end{equation*}
$$

where $M$ is the absolute magnitude of the star given in terms of the solar absolute magnitude $M_{\odot}$ by:

$$
\begin{equation*}
M=-2.5 \log _{10} L / L_{\odot}+M_{\odot} \tag{5}
\end{equation*}
$$

( $M_{\odot}$ is the absolute magnitude of the sun not to be confused with the solar mass). If we plug in $L$ for the patch of light in 1 sq . arcsec into the formula above and recognize that $\mu \equiv m$ the apparent magnitude of the square arcsecond patch in this context we determine:

$$
\begin{aligned}
\mu & =-2.5 \log _{10}\left(I d^{2} \delta \theta^{2}\right)+5 \log _{10} d-5+M_{\odot} \\
\mu & =-2.5 \log _{10} I-5 \log _{10}(\delta \theta)-5+M_{\odot} \\
\mu & =-2.5 \log _{10} I /\left(L_{\odot} p c^{-2}\right)+M_{\odot}+21.572
\end{aligned}
$$

Notice that the distance $d$ cancels so that the quantity $\mu$ is independent of distance and $I$ is measured in units of $\mathrm{L}_{\odot} \mathrm{pc}^{-2}$. The constant just comes from the size of an arcsecond in radians. (Note when the distance of galaxies becomes large - cosmological effects and the curved geometry of the universe become important and change the relationship between distance and surface brightness but the above formula is valid for nearby galaxies within a 100 Mpc or so).
As an example, the K-band central surface brightness of the galaxy is estimated to be $I_{0}=1208$ $\mathrm{L}_{\odot} \mathrm{pc}^{-2}$ with $M_{K}=3.28$ for the sun, so what would be the measured surface brightness for an external observer? Using the formula above,

$$
\begin{equation*}
\mu_{K}=-2.5 \log _{10}(1208)+3.28+21.572=17.15 \text { mag. per sq. arcsec } \tag{6}
\end{equation*}
$$

## $\underline{\text { Radial Surface Brightness Profiles for Galaxies }}$

Spiral galaxies are usually observed to have exponential radial profiles after correction for inclination given by:

$$
\begin{equation*}
I(R)=I_{o} \exp \left(-R / R_{d}\right) \tag{7}
\end{equation*}
$$

The total luminosity is given by $L=\int_{0}^{\infty} I(R) R d R$. Solving the integral gives $L_{t o t}=2 \pi R_{d}^{2} I_{o}$. We can also express this as:

$$
\begin{equation*}
\mu=\mu_{0}+\frac{2.5}{\ln 10}\left(\frac{R}{R_{d}}\right)=\mu_{0}+1.09\left(\frac{R}{R_{d}}\right) \tag{8}
\end{equation*}
$$

Elliptical galaxies tend to follow the $R^{1 / 4}$ or deVaucouleurs (1948) law:

$$
\begin{equation*}
I(R)=I_{o} \exp \left(-\left[R / R_{0}\right]^{1 / 4}\right) \tag{9}
\end{equation*}
$$

where $I_{o}$ is the central surface brightness and $R_{o}$ is a scale-length. This profile is a bit of a pain to deal with mathematically but using the substitution $u=\left(R / R_{0}\right)^{1 / 4}$ or $R=R_{0} u^{4}$ the total luminosity is given by:

$$
\begin{equation*}
L_{t o t}=8 \pi I_{0} R_{0}^{2} \int_{0}^{\infty} u^{7} \exp (-u) d u \tag{10}
\end{equation*}
$$

We can solve this integral using the identity:

$$
\begin{equation*}
\int_{0}^{u_{0}} u^{n} \exp (-u) d u=n!\left[1-\exp (-u)\left(1+u+u^{2} / 2!+\ldots+u^{n} / n!\right)\right] \tag{11}
\end{equation*}
$$

so that $L_{t o t}=8 \pi 7!I_{0} R_{0}^{2}$.
The $R^{1 / 4}$ law is usually characterized by the effective radius $R_{e}$ i.e. the radius that contains exactly half the total light. To compute this radius, we need to solve the equation:

$$
\begin{equation*}
2 \pi \int_{0}^{R_{e}} I(R) R d R=\frac{L_{t o t}}{2} \tag{12}
\end{equation*}
$$

The solution gives the result $\left(R_{e} / R_{o}\right)^{1 / 4}=7.669$. We also characterize the profile in terms of $I_{e}$, the surface brightness measured at the effective radius. With these changes the usual form seen in the astronomical literature is:

$$
\begin{equation*}
I(R)=I_{e} \exp \left\{-7.669\left[\left(R / R_{e}\right)^{1 / 4}-1\right]\right\} \tag{13}
\end{equation*}
$$

or

$$
\begin{align*}
\mu & =\mu_{e}-2.5(-7.669) / \ln 10\left[\left(R / R_{e}\right)^{1 / 4}-1\right]  \tag{14}\\
\mu & =\mu_{e}+8.3268\left[\left(R / R_{e}\right)^{1 / 4}-1\right] \tag{15}
\end{align*}
$$

$\underline{\text { Relationship between 2D and 3D profiles }}$
For spherical galaxies, the measured surface brightness represents the projection of the 3D radial profile of the stars. The luminosity density in 3D is usually given by the function, $j(r)$. We can determine the observed intensity by integrating along the line of sight viz.

$$
\begin{equation*}
I(R)=2 \int_{0}^{\infty} d z j(r) \tag{16}
\end{equation*}
$$

Since $r^{2}=R^{2}+z^{2}$ where $r$ is the spherical radius and $R$ is the cylindrical radius then:

$$
\begin{equation*}
I(R)=2 \int_{R}^{\infty} \frac{j(r) r d r}{\sqrt{r^{2}-R^{2}}} \tag{17}
\end{equation*}
$$

This can actually be inverted using an Abel integral identity through:

$$
\begin{equation*}
j(r)=-\frac{1}{\pi} \int_{r}^{\infty} \frac{d I}{d R} \frac{d R}{\sqrt{R^{2}-r^{2}}} \tag{18}
\end{equation*}
$$

This means in principle if we observe the 2 D profile, we can determine the 3 D profile and then have a handle of the gravitational field of the stars with some assumption of the mass to light ratio $M / L$ i.e. the 3D density is just $\rho(r)=(M / L) j(r)$. The deprojection of the deVaucouleurs law has been done but there is no simple analytical expression for $j(r)$. Hernquist (1990) noted that the density profile $\rho(r)=M a /\left[2 \pi r(r+a)^{3}\right]$ generated by the potential $\Phi(r)=-G M /(r+a)$ looks very much like a deVaucoleurs profile when projected to determine $I(R)$. The effective radius $R_{e}=1.82 a$.

Mass Modeling of Spiral Galaxies: Bulge-disk decomposition
The radial surface brightness profile of spiral galaxies appears to be a superposition of a bulge and disk component so total profiles are usually fit with the profile:


Figure 1: A mass model of the galaxy M31 with rotation curve (including dark matter) plus surface brightness (from Widrow and Dubinski 2005).

$$
\begin{equation*}
I(R)=I_{d}(R)+I_{b}(R) \tag{19}
\end{equation*}
$$

where $I_{d}(R)$ is usually modelled as an exponential disk and $I_{b}(R)$ is modelled as a deVaucouleurs law. Each model function has 2 parameters so the net fit has a total of 4 parameters. The general procedure is to take an image of galaxy in some bandpass like the B or K band and then fit isophotal contours to the time image to measure $I(R)$. A nonlinear least squares fit is then applied to the data to determine the parameters $I_{0}$ and $R_{d}$ for the disk and $I_{e}$ and $R_{e}$ for the bulge. In this way, we have separated out these 2 components and can consider them independently. We can then assume a mass to light ratio for both components and thus determine a physical surface density $\Sigma(R)$ and 3D density for the bulge, $\rho(r)$. Both of these physical mass densities can be converted into gravitational potentials by solving Poisson's equations and thus find the expected rotation curves. The solution for the disk is complex but solvable while a spherical bulge profile is simpler to deal with.

