MOND N-body: why $a = \sqrt{a_0 g_N}$ isn't enough

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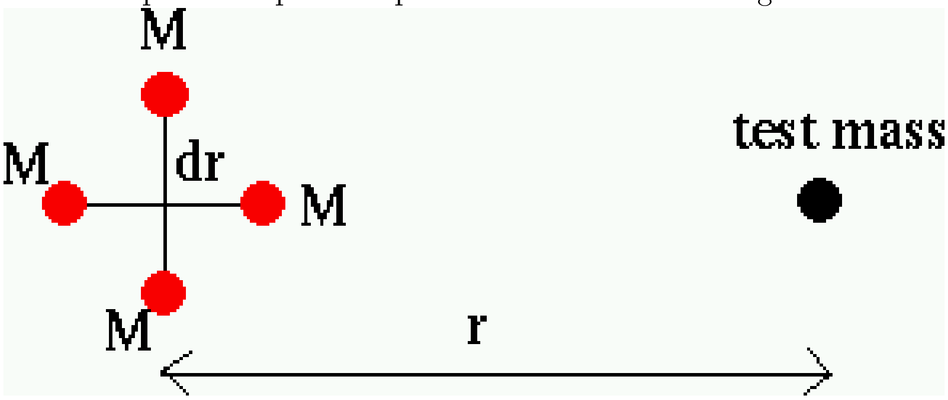
What would happen if you replaced the regular Newtonian force law in your N-body simulations with the MOND force law? – just about everyone I know has asked me this.

Well, it ain't that simple. Here's why:

Let's do the following thought experiment. Under Newtonian gravity, take a particle with mass $4M$ and calculate the acceleration on a test particle at distance r. We have

$$a = \frac{G(4M)}{r^2} = \frac{4GM}{r^2}$$

Okay, now break that particle up into 4 particles of mass M arranged like this:



Now calculate the acceleration:

$$a = \frac{GM}{(r+dr)^2} + \frac{GM}{(r-dr)^2} + \frac{2GM}{r^2+dr^2}$$

or

$$a = \frac{GM}{r^2}\left[\frac{1}{(1+x)^2} + \frac{1}{(1-x)^2} + \frac{2}{(1+x^2)}\right]$$

where $x \equiv \frac{dr}{r} << 1$.

Now, expand each of those terms as a series, and keep terms to x^2:

$$a = \frac{GM}{r^2}\left[(1-2x+3x^2) + (1+2x+3x^2) + 2(1-x^2)\right]$$

or

$$a = \frac{4GM}{r^2}(1+x^2) = \frac{4GM}{r^2} \text{ for } x << 1$$

Okay, this is good: if the separation is tiny, we get the same total force as we did when we had a single mass of $4M$.

Now, let's do the same thing under MOND. In the MOND limit, the usual force parameterization is $a = \sqrt{a_0 g_N} = \frac{\sqrt{GMa_0}}{r}$. So if the original mass is 4M, the acceleration is

$$a = \frac{\sqrt{G(4M)a_0}}{r} = \frac{2\sqrt{GMa_0}}{r}$$

Now let's break up the mass and add forces again.

$$a = \frac{\sqrt{GMa_0}}{r+dr} + \frac{\sqrt{GMa_0}}{r-dr} + \frac{2\sqrt{GMa_0}}{\sqrt{r^2+dr^2}}$$

or

$$a = \frac{\sqrt{GMa_0}}{r}\left[\frac{1}{1+x} + \frac{1}{1-x} + \frac{2}{\sqrt{1+x^2}}\right]$$

Again, expand each of those terms as a series, and keep terms to x^2:

$$a = \frac{\sqrt{GMa_0}}{r}\left[(1-x+x^2) + (1+x+x^2) + 2(1-\frac{x^2}{2})\right]$$

or

$$a = \frac{\sqrt{GMa_0}}{r}\left[4+x^2\right]$$

or, for $x << 1$,

$$a = \frac{4\sqrt{GMa_0}}{r}$$

which is twice the force from the original mass! Not good...

In doing N-body work, we assume we can represent mass coarsely – instead of resolving each star, we use massive particles which represent N stars. Under Newtonian mechanics that's fine - N particles of mass M exert the same force as 1 particle of mass NM. But that's not true in this parameterization of MOND, so simply replacing a with $a = \sqrt{a_0 g_N}$ won't work to do MOND N-body calculation. Note that this does not mean MOND is *wrong*, just that this kind of calculation won't work.