APPENDIX

Here we apply the formalism of analytic population synthesis (Tinsley 1972; Tinsley & Gunn 1976) to the case of constant star formation. This is useful for illustrating the effects of variable turn on times on the relative balance between the contributions of main sequence (MS) and post main sequence (PMS) stars, and the subsequent impact on the colors of a stellar system composed of many generations.

It is standard and convenient to adopt power law relations which describe the stellar population and stellar physics. These are of course approximations, and single slope power laws do not provide perfect fits for the entire range of stellar properties, which may also have other dependencies than those primarily assumed. This is, however, quite adequate for our current purposes.

We shall decompose the stellar birthrate function, which describes the stellar population, into a mass spectrum (the IMF) and a star formation rate (SFR). The IMF gives the relative number of stars of different masses,

$$\frac{dN}{dm} = \phi(m) = \phi_0 m^{-(1+x)}, \tag{A1}$$

which is assumed to be valid between some limiting upper (m_u) and lower (m_ℓ) mass. The precise values of x and m_u are controversial, even though m_u may only be determined in a probabilistic way from the distribution defined by x. The precise values of the parameters are not important to what follows, though we will assume that x is constant and has an astrophysically plausible value (i.e., low mass stars are more frequent than high mass stars, so $x > 0$). We consider SFRs of the form illustrated in Figure 9,

$$\psi_j(t) = \psi_0 u(t_j), \tag{A2}$$

where $u(t_j)$ is the step function. Thus star formation model j is characterized by the turn on time t_j.

The stellar physics is encapsulated in the main sequence mass–luminosity and mass–lifetime relations. Respectively, these are

$$\ell(m) = \ell_0 m^{\xi_k} \tag{A3}$$

and

$$m(t) = \mathcal{M}_0 t^{-\theta}. \tag{A4}$$

Note that we allow the slope of the mass–luminosity relation to depend on the observational bandpass k. We will assume that the luminosity of post main sequence stars is adequately modeled by a constant luminosity $\mathcal{L}^{\rm PMS}$ for each post main sequence phase over the (relatively short) lifetime of that phase $\tau^{\rm PMS}$.

Combining these allows us to write a simple expression for the luminosity produced by a single generation of stars. For the main sequence, this is simply the convolution of the IMF and mass–luminosity relation:

$$L_i^{\rm MS} = \int_{m_\ell}^{m_\tau} \phi(m)\ell(m)dm. \tag{A5}$$

The upper limit is set by the main sequence turn off mass m_τ, so the value of m_u is irrelevant. Similarly, since we are interested in colors and not mass to light ratios, we can set $m_\ell \rightarrow 0$, as low mass stars may represent much mass but produce very little light. Substituting into equation (A5) the relations defined above, we have

$$L_i^{\rm MS} = \phi_0\ell_0 \int_0^{m_\tau} m^{\xi_k - 1 - x}dm, \tag{A6}$$

which gives

$$L_i^{\rm MS} = \frac{\phi_0\ell_0}{\xi_k - x} m_\tau^{\xi_k - x}. \tag{A7}$$

To get the contribution from all generations, we must convolve equation (A7) with the weighted contribution of each, so

$$L_{\rm (total)}^{\rm MS} = \sum w_i L_i = \int w(t)L(t)dt. \tag{A8}$$

Here, L depends on t through equation (A4), and the weights are just the normalized SFRs $w_j(t) = \psi_j(t)/<\psi_j>$. This gives

$$L^{\rm MS} = \frac{\phi_0\ell_0}{<\psi_j>(\xi_k - x)} \int_0^{t_0} \psi(t) m_\tau(t)^{\xi_k - x}dt, \tag{A9}$$

where t_0 is the present epoch. Replacing m_τ with equation (A4) and allowing the chosen step function form of the SFR (equation A2) to operate on the limits of integration,

$$L_{jk}^{\rm MS} = \frac{\phi_0\ell_0\psi_0\mathcal{M}_0^{\xi_k - x}}{<\psi_j>(\xi_k - x)} \int_{t_j}^{t_0} t^{-\theta(\xi_k - x)}dt. \tag{A10}$$

Thus we have an expression for the main sequence k-band luminosity of a composite stellar population specified by star formation history j:

$$L_{jk}^{\rm MS} = \frac{\phi_0 \ell_0 \psi_0 \mathcal{M}_0^{\xi_k - x}}{<\psi_j>(\xi_k - x)(\theta x - \theta \xi_k + 1)} t^{\theta x - \theta \xi_k + 1} \Bigg|_{t_j}^{t_0}. \tag{A11}$$

Note that the limits do matter as t_j specifies the star formation history.

The numbers of post main sequence stars from any generation follows directly from the rate at which stars are turning off the main sequence and the amount of time they spend in each PMS phase, $\tau^{\rm PMS}$:

$$N_i^{\rm PMS} = \frac{dN}{dm}\left|\frac{dm}{dt}\right|_{m_\tau} \tau^{\rm PMS} = -\phi(m_\tau)\frac{dm_\tau}{dt}\tau^{\rm PMS}. \tag{A12}$$

Substituting the appropriate expressions, this becomes

$$N_i^{\rm PMS} = \phi_0 \tau^{\rm PMS} \mathcal{M}_0^{-x} \theta t^{\theta x - 1}, \tag{A13}$$

which when multiplied by the average luminosity of each phase gives the total luminosity produced by that branch,

$$L_i^{\rm PMS} = \phi_0 \tau^{\rm PMS} \mathcal{L}^{\rm PMS} \mathcal{M}_0^{-x} \theta t^{\theta x - 1}. \tag{A14}$$

This can be weighted by the star formation history as well,

$$L_{jk}^{\rm PMS} = \frac{\psi_0}{<\psi_j>}\phi_0 \tau^{\rm PMS} \mathcal{L}^{\rm PMS} \mathcal{M}_0^{-x} \theta \int_{t_j}^{t_0} t^{\theta x - 1} dt, \tag{A15}$$

which gives simply

$$L_{jk}^{\rm PMS} = \frac{\psi_0}{<\psi_j>}\frac{\phi_0}{x \mathcal{M}_0^x} \tau^{\rm PMS} \mathcal{L}^{\rm PMS} t^{\theta x} \Bigg|_{t_j}^{t_0}. \tag{A16}$$

The total luminosity contributed by each post main sequence phase is about equally divided between the giant branch, the HB, and the AGB (Renzini & Buzzoni 1986). As long as this is true (see §5.3), the light produced by PMS stars will always be redder than that from the MS since the "average" PMS star is closer to the giant branch than the horizontal branch (which itself is often red). Dividing equation (A16) by equation (A11) gives the ratio of PMS to MS luminosity:

$$\frac{L_{jk}^{\rm PMS}}{L_{jk}^{\rm MS}} = \frac{\tau^{\rm PMS} \mathcal{L}^{\rm PMS}}{\mathcal{M}_0 \ell_0} \frac{(\xi_k - x)(\theta x - \theta \xi_k + 1)}{x} \frac{t^{\theta x}|_{t_j}^{t_0}}{t^{\theta x - \theta \xi_k + 1}|_{t_j}^{t_0}}. \tag{A17}$$

This gives the relative weight of the predominantly red and blue components, and hence some indication of the composite color. Assuming that the IMF is the same, and that differences in metallicity do not excessively alter the composite spectral energy distribution of the various phases, we can cancel all but the temporal terms, and compare the effects of different star formation histories on different filters. For the cases illustrated in Figure 9, $j = \{\mathrm{A}, \mathrm{B}\}$, $t_{\mathrm{B}} > t_{\mathrm{A}}$, and filters $k = \{B, I\}$,

$$\left. \frac{\frac{L^{\mathrm{PMS}}}{L^{\mathrm{MS}}}|_{\mathrm{A}}}{\frac{L^{\mathrm{PMS}}}{L^{\mathrm{MS}}}|_{\mathrm{B}}} \right|_{B} < \left. \frac{\frac{L^{\mathrm{PMS}}}{L^{\mathrm{MS}}}|_{\mathrm{A}}}{\frac{L^{\mathrm{PMS}}}{L^{\mathrm{MS}}}|_{\mathrm{B}}} \right|_{I} \tag{A18}$$

as long as $\xi_B > \xi_I$, which is guaranteed by the main sequence temperature–luminosity relation. Note that differences in the absolute rates of star formation are irrelevant, since ψ_0 has already been canceled out of equation (A17) — only the shape of the star formation history matters. Equation (A18) simply states that the older population (A) will produce relatively more red light. This arises from the increasing contribution of (red) post main sequence stars to the total luminosity as time goes on, i.e., the increasingly populace giant branch (cf. Renzini & Buzzoni 1986). Hence the blue colors and lack of diffuse red disks in LSB galaxies can result from late collapse and ignition of star formation.