Cosmological Overdensities & Enclosed Mass: Relating $R_\Delta$, $M_\Delta$, and $V_\Delta$

It is conventional in cosmology to refer to structures by the density contrast they represent with respect to the critical density of the universe. The mass enclosed within a radius encompassing the over-density $\Delta$ is

$$M_\Delta = \frac{4\pi}{3} \Delta \rho_{\text{crit}} R_\Delta^3.$$  

With the definition of critical density

$$\rho_{\text{crit}} = \frac{3H_0^2}{8\pi G},$$

this becomes

$$M_\Delta = \frac{\Delta}{2G} H_0^2 R_\Delta^3.$$  

By the same token, the circular velocity of a tracer particle at $R_\Delta$ is

$$V_\Delta^2 = \frac{GM_\Delta}{R_\Delta}.$$  

Consequently,

$$M_\Delta = (\Delta/2)^{-1/2} (GH_0)^{-1} V_\Delta^3.$$  

The choice of reference $\Delta$ is somewhat arbitrary. For the current best-fit LCDM cosmology, the "virial" radius within which the mass has had time to settle down occurs around $\Delta \approx 100$. (The exact virial value of $\Delta$ depends weakly on the the cosmological parameters. The definition of virial radius, while formally meaningful, is nevertheless somewhat arbitrary as the mass profiles of dark matter halos merge smoothly into their surroundings.) For a Hubble constant $H_0 = 72$ km s$^{-1}$ Mpc$^{-1}$,

$$M_{100} = (4.6 \times 10^5$ km$^{-3}$ s$^{-3}$ M$_\odot)V_{100}^3.$$  

for $\Delta = 200$, as conventionally

(if not quite correctly) used,

$$M_{200} = \left(3.3 \times 10^5 \left(\frac{k_{\text{m}}}{5}\right) M_\odot\right) V_{200}^3.$$  

1
**NFW Density Profile**

**Halo only**


\[
\rho(r) = \frac{\rho_s r_s^3}{r(r + r_s)^2},
\]

This integrates to a mass profile

\[
M(r) = 4\pi \rho_s r_s^3 \left[ \ln(1 + x) - \frac{x}{1 + x} \right] = M_* \left[ \ln(1 + x) - \frac{x}{1 + x} \right],
\]

where \( x = r/r_s \) and \( M_* = 4\pi \rho_s r_s^3 \). For small \( x \),

\[
M(x) \simeq M_* \left[ \frac{x^2}{2} - \frac{2x^3}{3} + \frac{3x^4}{4} \right] \simeq 2\pi \rho_s r_s r^2.
\]

The gravitational potential is the work done

\[
\Phi(r) = -\int_r^{\infty} \frac{GM(r')}{r'^2} \, dr'.
\]

Inserting the above expression for \( M(r) \), we obtain

\[
\Phi(r) = -\frac{GM_* \ln(1 + x)}{r_s x}.
\]

Notice that this does tend to zero as \( r \to \infty \) even though the total mass is logarithmically divergent. At small radii

\[
\Phi(r) \simeq -\frac{GM_*}{r_s} \left[ 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} \right]
\]

The radial acceleration is

\[
\frac{d\Phi}{dr} = \frac{GM_*}{r_s^2} \left[ \frac{1}{x(1 + x)} - \frac{\ln(1 + x)}{x^2} \right],
\]

and the circular speed is

\[
V^2(r) = \frac{GM_*}{r} \left[ \ln(1 + x) - \frac{x}{1 + x} \right].
\]

Since

\[
\kappa^2 = 2\Omega \left( \Omega + \frac{dV}{dr} \right) = \frac{2V^2}{r^2} + \frac{1}{r} \frac{dV^2}{dr},
\]

we find

\[
\kappa^2(r) = \frac{GM_*}{r^3} \left[ \ln(1 + x) - \frac{x}{1 + x} + \frac{x^2}{(1 + x)^2} \right].
\]
NFW introduce the "virial radius" \( r_{200} \), inside of which the average density is \( 200\rho_{\text{crit}} \), with
\[
\rho_{\text{crit}} = 3H_0^2/(8\pi G) = 1.88 \times 10^{-29} h^2 \text{ g cm}^{-3} = 2.76 \times 10^{-7} h^2 \text{ M}_\odot \text{ pc}^{-3}.
\]
They further define \( c = r_{200}/r_s \). Now
\[
V_{200}^2 = \frac{GM_{200}}{r_{200}} = \frac{G}{c r_s} c r_s \frac{3H_0^2}{8\pi G}, \quad \text{or} \quad V_{200} = 10c r_s H_0,
\]
and
\[
\frac{\rho_s}{\rho_{\text{crit}}} = \frac{200c^3}{3[\ln(1+c) - c/(1+c)]} = \frac{200}{3} [A(c)]^2.
\]
For small values of \( c \) we have \([A(c)]^2 \approx 6c/(3 - 2c)\). (For those interested in trivia, \( \rho_s = \rho_{\text{crit}} \) when \( c \approx 3/402 \).) The function \( A(c) \) looks like:

![Graph of A(c)](image)

From this, we also get
\[
V(x) = V_{200} \left[ \left( \frac{c}{x} \right) \ln(1+x) - x/(1+x) \right]^{1/2},
\]
which has a maximum at \( x \approx 2.162582 \) with the value
\[
V_{\text{max}} \approx 0.465V_{200} \left[ \frac{c}{\ln(1+c) - c/(1+c)} \right]^{1/2} = \frac{0.465V_{200}}{c} A(c) = 4.65 r_s H_0 A(c).
\]
The average density inside \( r_s \) is
\[
\bar{\rho}_1 = \frac{3V^2(1)}{4\pi G r_s^2} = \frac{3V_{200}^2}{4\pi G r_s^2} \frac{A^2(c)}{c^3} [\ln 2 - 1/2] = \frac{3 \times 0.193 A^2(c)}{4\pi G} 100 H_0^2.
\]
In terms of the critical density, this is
\[
\frac{\bar{\rho}_1}{\rho_{\text{crit}}} = \frac{3 \times 19.3 A^2(c)}{4\pi G} H_0^2 \times \frac{8\pi G}{3H_0^2} = 38.6 A^2(c),
\]
which is less than \( \rho_s \) because \( \rho(r_s) = \rho_s/4 \) — in fact, \( 0.58 \rho_s \approx \bar{\rho}_1 \approx 2.32 \rho_s \).
Einasto halo

A better fitting function for simulated halos than NFW
at the expense of an extra parameter

Merritt et al. (2006):

\[ \rho(r) = \rho_e \left[ \left( \frac{r}{r_e} \right)^{1/n} - 1 \right] \]

If that looks familiar, it is because the Einasto profile is the 3D equivalent of
the Sersic profile used for fitting projected surface densities.

\[ \rho_e = \text{density at radius } r_e \]
that contains \( \frac{1}{2} \) of the total mass

\( d_n \) is a hassle to obtain, but is well approximated by

\[ d_n \approx 3n - \frac{1}{3} + \frac{0.0079}{n} \quad \text{for } n > \frac{1}{2} \]

Simulated halos have \( 4.6 < n < 8.2 \)

In the notation of Navarro et al. (2004), \( \alpha = \frac{1}{n} \)

who found the midpoint \( \alpha \approx 0.17 \)

The variable \( n \) is sometimes involved to explain cores
in real galaxies, but this is not correct.

Both the change in slope and its location
at very small radii) fail to explain observations.
It is too small an effect - really just a tweak
to NFW
Einasto halos do have the nice property of a finite total mass

\[ M(<r) = 4\pi \int_0^r \frac{r^3 \rho e^{\frac{3}{3-n}} d\rho}{\Gamma(3-n)} \]

where \( \gamma \) is the incomplete gamma function.

Integrating \( M(r) \) to \( \infty \) turns \( \gamma \) into \( \Gamma \)

\[ M_{bh} = 4\pi \int_0^\infty r^3 \rho e^{\frac{3}{3-n}} \frac{d\rho}{\Gamma(3-n)} \]


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Empirical DM Halo

McGaugh et al. (2007)
Walker et al. (2009)

just fit the portion of the data attributable to dark matter

Adopt \( M_\odot/L \) & choose that which minimizes scatter - in both BTFK and \( V_{\odot}(R) \), as it happens. Get

\[ \log_{10} V(R) = 1.47 + 0.15 \log R - 0.19 \frac{1}{2} \log R \quad (V \text{ in } \text{km s}^{-1}) \]

slope \( \frac{1}{2} \) fitted to NFW value (best fit 0.49)

this can also be expressed in terms of the enclosed mass

\[ M(R) = \frac{200}{-120} \left( \frac{R}{\text{pc}} \right)^2 M_\odot \quad (R \text{ in pc}) \]

note: this is not the same as \( \rho_{\odot} \) which increases linearly

or

\[ g_{\odot} = 3 \times 10^{-14} \text{ m s}^{-2} \]